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Special projective varieties of higher dimension

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Contents

Introduction

In the last decades one of the most important theories in Algebraic Geometry was introduced: in his fundamental papers [Mo79] and [Mo82], S. Mori studied smooth 3-folds X with non numerically effective canonical bundle, i.e. 3-folds admitting a "negative" curve, which is a curve C such that $K_X \cdot C < 0$.

Among the new results proved by S. Mori, we recall the fundamental *Cone Theorem* and *Contraction Theorem*. The *Cone Theorem* gives a description of the *Kleiman-Mori cone* of X, which is the closed convex cone generated by effective 1-cycles: it states that all the "negative" curves are in a part of this cone that turns out to be locally polyhedral, and whose countably many rays are generated by rational curves. According to the *Contraction Theorem*, each ray R (or face F) in the polyhedral part of the cone can be contracted. More precisely, associated to any ray R (or face F), there exists a proper surjective map $\Phi\colon X \longrightarrow Y$ onto a normal variety, which exactly contracts all the curves with numerical class in R (or F), whose fibers are connected, and such that the anticanonical divisor of X is Φ-ample. The map Φ is usually called a *Fano-Mori contraction*, or an *extremal contraction*; it is said *elementary* if it is associated to an extremal ray.

We note that in general an extremal contraction leads to a singular variety. Many people worked on these varieties; among the others we recall M. Reid, who found the "good" singularities to work with in his papers [Re80] and [Re83].

An important step in the theory is the generalization of these theorems of Mori to higher dimensional varieties. This has been done by Y. Kawamata, V.V. Shokurov and others in different papers, among which we recall [Ka84a], [Ka84b] and [Sh85]. Moreover their proofs allowed some mild singularities on the variety, too.

With all these results, it has been possible to develop a program for the construction of "minimal models" of class of varieties (we recall that a variety X is called *minimal* if its anticanonical bundle is numerically effective), called "Minimal Model Program" ("MMP"), or "Mori theory", whose aim is to choose a "simple" object inside a birational class of varieties, via the contractions of the extremal rays.

Along the program a new problem arises: it can happen that an extremal ray contracts a subset of codimension at least 2. In the 3-dimensional case, with the complete description of terminal singularities in dimension 3 of [Mo85] and [Re87] and the construction in [Ka88], S. Mori solved this problem (and therefore the MMP), in his paper [Mo88].

However in general it is difficult and still an open problem even in dimension 3 to follow the birational map from a variety X to $\widetilde{X}$. In the case $\dim X = 3$, with the help of a polarization (i.e. the existence of an ample line bundle H) of "small degree" (i.e. small H^3), a special case of minimal model (called *$\sharp$-MMP*) can be carried on, as shown by M. Mella in [Me02]. Since the output is always a uniruled variety polarized by an ample line bundle, a natural problem is to look for a condition under which it is "uniruled by lines" with respect to the ample line bundle that gives the polarization. We will deal with this problem in the last chapter.

Another famous theorem of S. Mori in [Mo79] states that, given a pair $(X, \mathcal{E})$ where $\mathcal{E}$ is an ample vector bundle on the projective manifold X, if $\mathcal{E}$ is the tangent bundle of X, then X is the projective space. M. Andreatta and J.A. Wiśniewski, in [AnWi01], have shown that this result holds even more generally, when $\mathcal{E}$ is just a subsheaf of the tangent bundle of X.
We will deal with the classification of pairs $(X, \mathcal{E})$, where X is a non minimal projective manifold and $\mathcal{E}$ is an ample vector bundle, in the second chapter, under suitable assumptions on the *nef value* $\tau := \min\{t \in \mathbb{R} : K_X + t \det \mathcal{E} \text{ is nef}\}$ of the pair $(X, \det \mathcal{E})$ (we refer to [KaMaMa87] for all details), describing the morphism associated to an extremal ray in the extremal face $F(\mathcal{E}) := \{C \in \overline{NE}(X) \mid (K_X + \tau \det \mathcal{E}) \cdot C = 0\}$.

In general the study and the classification of *polarized varieties*, i.e. pairs (X, L) given by a variety X and a very ample line bundle L on X (i.e L is a hyperplane in a given embedding of X), is the object of the "Adjunction Theory", for which we refer to [BeSo95]. This theory, developed by A.J. Sommese, P. Ionescu, T. Fujita, L. Bădescu, M.C. Beltrametti and others, basically works reducing the study of the variety X to a hyperplane section of X, via the adjunction formula.
As an extension of this theory, the "Generalized Adjunction" studies pairs $(X, \mathcal{E})$ given by a manifold X and an ample vector bundle $\mathcal{E}$ on X, called *polarized varieties* as well.

The classification of polarized varieties $(X, \mathcal{E})$ is strictly connected to the study of the Kleiman-Mori cone of X. Moreover, given a pair $(X, \mathcal{E})$, a natural problem to investigate is the relation between X and an ample section $Z = (s)_0$, with $s \in \Gamma(\mathcal{E})$, of codimension r in X. We recall that, if the dimension of Z is at least 3, we can see the Kleiman-Mori cone of Z as a subset of the Kleiman-Mori cone of X. Therefore, in the spirit of Mori theory, the relations between the cones give a first step to compare the structures of X and Z. These relations have been studied by T. de Fernex and A. Lanteri in [dFLa99] and by M. Andreatta and G. Occhetta in [AnOc99] and [AnOc02a]; in the third chapter we will give some new results and some applications.

Going into the details: in the first chapter we recall some fundamental results of the theory of extremal rays and extremal contractions, together with some results on Fano manifolds.

In the second chapter we deal with polarized varieties $(X, \mathcal{E})$, given by a smooth complex projective variety X of dimension n and an ample vector bundle $\mathcal{E}$ of rank r on X.
We assume that X is not minimal and we give a classification of the polarized varieties $(X, \mathcal{E})$, with suitable assumptions on the *nef value* τ of the polarized variety $(X, \det \mathcal{E})$, describing the morphism associated to an extremal ray in the extremal face $F(\mathcal{E})$. This classification, contained in [AnNopr], will be done for all $\tau \geq \frac{n-2}{r}$. If $r = 1$ this is a classical start up; for a complete survey we refer the reader to [BeSo95]. If $1 = \tau (\geq \frac{n-2}{r})$ this subject was developed by many authors in the following series of papers: [Pe90], [YeZa90], [Fu90b], [Pe91], [AnBaWi92], [LaMa95], [Ma95] and [AnMe97]. Building on the above quoted papers in [Oh99] and [Ohpr] M. Ohno classified the pairs for $\tau \geq \frac{n-2}{r}$, with the further hypothesis $\tau \geq 1$. However his proofs are different and much longer. Moreover we improve his bound on the nef value τ, proving that $\tau \leq \frac{l(R)}{r}$, where $l(R)$ is the length of the extremal ray we are dealing with.

In the third chapter given a polarized variety $(X, \mathcal{E})$, we compare the geometry of X with the geometry of an ample section, $Z = (s)_0$, with $s \in \Gamma(\mathcal{E})$, of codimension r in X. In particular we compare the two Kleiman-Mori cones, giving examples in which they are different and conditions under which the significant part of the cones (i.e. the negative part) is the same. Moreover we study the case in which Z does not have extremal rays but its canonical divisor is not ample and we classify the polarized varieties $(X, \mathcal{E})$ such that Z is a smooth surface of Kodaira dimension 0 or 1. We also study polarized varieties (X, L), where L is an ample line bundle on X and $Z \in |L|$ is a special Fano manifold: indeed we deal with the case in which Z is either a Mukai manifold, or a Fano manifold of index $\dim Z - 3$, or $-K_Z = \det \mathcal{V}$ for some ample vector bundle $\mathcal{V}$ on Z of rank $r \geq \dim Z - 2$.
The main part of this chapter contains the results of [AnNoOcpr].

In the last chapter, which is part of a work in progress in collaboration with A. Sarti and A.L. Knutsen ([KnNoSa]), we deal with some Mori fiber spaces; in particular we give some bounds on the degree of a polarized terminal $\mathbb{Q}$-factorial uniruled 3-fold to find that the 3-fold is uniruled by lines and we look for a sharp bound. For this purpose we study in details sections of 3-dimensional *Terminal Veronese Fibrations*, i.e. sections of terminal $\mathbb{Q}$-factorial uniruled 3-folds which are fibrations over a smooth curve with general fiber a Veronese surface.

Chapter 1

Background material

The aim of this chapter is to collect all the definitions and the main results which we will use throughout the thesis. In particular we recall the fundamental "Cone Theorem" (1.3.1) and "Contraction Theorem" (1.3.12) of Mori theory and some results (1.5.5 and 1.5.6) on Fano varieties due to J.A. Wiśniewski.

1.1 Preliminaries

Let X be a normal complex projective variety of dimension n.
We will use the following notations:
$Z_{n-1}(X) :=$ the group of Weil divisors on X;
$Z_1(X) :=$ the free abelian group generated by irreducible reduced curves on X;
$\mathrm{Div}(X) :=$ the group of Cartier divisors on X;
$\mathrm{Pic}(X) :=$ the group of line bundles on X.
Let C be a complete curve on X and $f\colon \widetilde{C} \longrightarrow C$ its normalization. For every $D \in \mathrm{Pic}(X)$ we define an intersection number $D \cdot C := \deg_{\widetilde{C}} f^*D$; this intersection product induces a bilinear form

$$(\cdot)\colon \mathrm{Pic}(X) \times Z_1(X) \longrightarrow \mathbb{Z}$$

which induces on both groups an equivalence relation, which we will call *numerical equivalence* and denote by $\equiv$. Now we define

$$N_1(X) = \frac{Z_1(X)}{\equiv} \otimes \mathbb{R}, \quad N^1(X) = \frac{\mathrm{Pic}(X)}{\equiv} \otimes \mathbb{R}.$$

These two groups are canonically dual via $(\cdot)$; by the Neron-Severi theorem, we have $\dim_{\mathbb{R}} N_1(X) = \dim_{\mathbb{R}} N^1(X) < \infty$; we denote by $\rho(X)$ the dimension of these two groups and we call it the *Picard number* of X.

Inside $N_1(X)$ we consider the convex cone generated by the effective 1-cycles

$$NE(X) := \left\{ C \in N_1(X) \mid C = \sum a_i C_i, \text{ with } a_i \in \mathbb{R}_{\geq 0} \text{ and } C_i \text{ irreducible curves} \right\}$$

and the cone

$$\overline{NE}(X) := \text{the closure of } NE(X) \text{ in the real topology of } N_1(X)$$

which is called the *Kleiman-Mori cone* of X.

We will also use the following notation:
$\overline{NE}_{D\geq 0}(X) := \{C \in \overline{NE}(X) \mid D \cdot C \geq 0\}$ with $D \in N^1(X)$; in the same way we define $\overline{NE}_{D>0}(X)$, $\overline{NE}_{D\leq 0}(X)$ and $\overline{NE}_{D<0}(X)$.
We will denote by K_X the canonical divisor of the variety X.

Definition 1.1.1. A Cartier divisor D on a normal projective X is called *ample* if some positive multiple mD of D is *very ample*, i.e. mD gives an embebbing $\varphi\colon X \hookrightarrow \mathbb{P}^n$ in a projective space.

We have the following numerical characterization of ampleness due to S.L. Kleiman [Kl66]:

Theorem 1.1.2. (Kleiman's ampleness criterion) *Let X be a (projective) variety. $H \in \mathrm{Pic}(X)$ is ample if and only if its numerical class $[H] \in N^1(X)$ is positive on $\overline{NE}(X) \setminus \{0\}$.*

This criterion naturally leads to the following

Definition 1.1.3. A Cartier divisor D on a normal projective variety X is called *numerically effective* (*nef*) if $D \cdot C \geq 0$ for all $C \in \overline{NE}(X)$, or, equivalently, if $D \cdot C \geq 0$ for all curves C in X.

Definition 1.1.4. A nef Cartier divisor D on a normal projective n-dimensional variety X is called *big* if $D^n > 0$.

Moreover we will need the following

Definition 1.1.5. An element of $Z_{n-1}(X)_{\mathbb{Q}} := Z_{n-1}(X) \otimes \mathbb{Q}$ (resp. of $\mathrm{Div}(X)_{\mathbb{Q}} := \mathrm{Div}(X) \otimes \mathbb{Q}$) is called a $\mathbb{Q}$-*divisor* (a *Cartier* $\mathbb{Q}$-*divisor*). Two elements $D_1, D_2 \in Z_{n-1}(X)_{\mathbb{Q}}$ are called $\mathbb{Q}$-*linearly equivalent*, $D_1 \sim_{\mathbb{Q}} D_2$, if there exists an integer $m \in \mathbb{N}$ such that $mD_1, mD_2 \in Z_{n-1}(X)$ and $mD_1 \sim mD_2$, where $\sim$ denotes the linear equivalence between Cartier divisors.

1.2 Singularities

In this section we simply recall the types of singularities we will find through the thesis.

Definition 1.2.1. Let X be a normal variety. X is $\mathbb{Q}$-*factorial* if every $\mathbb{Q}$-divisor is $\mathbb{Q}$-Cartier. X is called $\mathbb{Q}$-*Gorenstein* if there exists an integer $m \in \mathbb{N}$ such that $mK_X \in \mathrm{Div}(X)$; X is called *Gorenstein* if $K_X \in \mathrm{Div}(X)$; X is called s-*Gorenstein* if $s := \min\{t \in \mathbb{N} \mid tK_X \in \mathrm{Div}(X)\}$.

Definition 1.2.2. A normal variety X is said to have *terminal singularities* (resp. *canonical singularities*, *log terminal singularities*) if the following conditions hold:

1. X is $\mathbb{Q}$-Gorenstein;
2. there exists a resolution of singularities $f\colon Y \longrightarrow X$ such that $K_Y = f^*K_X + \sum a_i E_i$, with $a_i \in \mathbb{Q}$ and $a > 0$ (resp. $a_i \geq 0$, $a_i > -1$), where the E_i are the exceptional divisors for f.

Remark 1.2.3. The coefficients a_i are called *discrepancies.* They do not depend on the resolution; if $K_X \in \mathrm{Div}(X)$, then the $a_i \in \mathbb{Z}$. Moreover if X has terminal (or canonical or log terminal) singularities, the second condition is satisfied for any resolution.

Remark 1.2.4. If X is a surface, then X has canonical (resp. terminal) singularities if it has at most rational double points (resp. smooth points); in particular terminal singularities are smooth in codimension 2.

1.3 Extremal rays and contractions

In this part we collect some fundamental results of Mori theory.
The first result, the *Cone Theorem*, gives a description of the Kleiman-Mori cone of a log terminal variety X in terms of convex geometry. It is essentially due to S. Mori, who gave a proof (based on a argument of reduction to characteristic p) in the nonsingular case in [Mo82]. The extension to the singular case is due to Y. Kawamata (in [Ka84b]), V.V. Shokurov (in [Sh85]) and others.

Theorem 1.3.1. (Cone Theorem) *Let X be a log terminal variety; then*

$$\overline{NE}(X) = \overline{NE}_{K_X \geq 0}(X) + \sum R_j,$$

where $R_j := \mathbb{R}^+[C_j] := \{D \in NE(X) \mid D \equiv \lambda C_j,\ \lambda \in \mathbb{R}^+\}$ *for some irreducible reduced rational curve* $C_j \in Z_1(X)$ *such that* $0 < -K_X \cdot C_j \leq n+1$*. Such a curve is called an* extremal curve.
Moreover the R_j *are discrete in the space* $\{Z \in N_1(X) \mid K_X \cdot Z < -\epsilon\}$ *for all* $\epsilon > 0$*.*

Definition 1.3.2. A face in the negative part (with respect to K_X) of the Kleiman-Mori cone is called an *extremal face* of X. The R_j are called *extremal rays* of X.

Remark 1.3.3. We note that the *Cone Theorem* shows that, if the negative part of the Kleiman-Mori cone of the variety X is not empty, then it is locally polyhedral and it is spanned by countably many extremal rays.

The following is a fundamental result due to Y. Kawamata; for its proof we refer to [Ka84b], [KaMaMa87] and [ClKoMo].

Theorem 1.3.4. (Rationality Theorem) *Let* X *be a log terminal variety and* $H \in \mathrm{Div}(X)$ *an ample divisor; if* K_X *is not nef, then*

$$r := \max\{t \in \mathbb{R} \mid H + tK_X \text{ is nef}\}$$

is a rational number.

Corollary 1.3.5. *Let* F *be an extremal face of a log terminal variety* X*. Then there exists a nef divisor* H *on* X *such that*

1. $F = \{Z \in \overline{NE}(X) \mid H \cdot Z = 0\}$*;*

2. *the divisor* $mH - K_X$ *is ample for all integers* $m \gg 0$*.*

Definition 1.3.6. The nef divisor H is called a *good supporting divisor* for the face F.

The following theorem was proved by Y. Kawamata in [KaMaMa87] and V.V. Shokurov in [Sh85].

Theorem 1.3.7. (Base Point Free Theorem) *Let* X *be a normal projective variety with only* $\mathbb{Q}$*-factorial and terminal singularities; let* D *be a nef Cartier divisor on* X *such that* $aD - K_X$ *is ample for some* $a \in \mathbb{N}$*. Then* $|mD|$ *is base point free for all integers* $m \gg 0$*.*

1.3.8. As a consequence of 1.3.5 and 1.3.7, for $m \gg 0$ the linear system $|mH|$ gives a morphism $\varphi_{|mH|}\colon X \longrightarrow \mathbb{P}(H^0(X, mH))$.
We consider the Stein factorization of the morphism $\varphi_{|mH|}$:

$$\begin{tikzcd} X \arrow[rr, "\varphi_{|mH|}"] \arrow[dr, "\Phi"'] & & \mathbb{P}(H^0(X, mH)) \\ & Y \arrow[ur, "p"'] & \end{tikzcd}$$

where Φ has connected fibers, Y is a normal variety and p is a finite morphism.

We recall now the following

Definition 1.3.9. A *contraction* is a proper map $\Phi\colon X \longrightarrow Y$ of normal irreducible varieties with connected fibers. The map Φ is *birational* if $\dim Y = \dim X$, is of *fiber type* if $\dim Y < \dim X$. The *exceptional locus* $E(\Phi)$ is the smallest subset of X such that Φ is an isomorphism on $X \setminus E(\Phi)$.

Definition 1.3.10. A contraction Φ of a manifold X is called a *Fano-Mori contraction* if the anticanonical divisor $-K_X$ is Φ-ample. The contraction Φ is *elementary* if $\mathrm{Pic}(X)/\Phi^*(\mathrm{Pic}(Y)) \cong \mathbb{Z}$. An elementary contraction is called *divisorial* (resp. *small*) if it is birational and its exceptional locus has codimension 1 (resp. ≥ 2).

Proposition 1.3.11. *[KaMaMa87, Proposition 5.1.6] Let $\Phi\colon X \longrightarrow Y$ be an elementary Fano-Mori contraction. If Φ is divisorial, then the exceptional locus is a prime divisor.*

As a consequence of 1.3.5, 1.3.7 and 1.3.8, we have the following theorem, which is a starting point of the MMP.

Theorem 1.3.12. (Contraction Theorem) *Let X be a log terminal variety. Let $H \in \mathrm{Div}(X)$ be a nef divisor such that $F := H^{\perp} \cap \overline{NE}(X) \setminus \{0\}$ is entirely contained in $\left\{Z \in N_1(X) \mid K_X \cdot Z < 0\right\}$. Then there exists a projective morphism $\Phi\colon X \longrightarrow Y$ onto a normal projective variety, which is characterized by the following properties:*

1. *for any irreducible curve C on X, $\Phi(C)$ is a point if and only if $H \cdot C = 0$;*

2. *Φ has only connected fibers;*

3. $H = \Phi^* A$ *for some ample Cartier divisor* $A \in \mathrm{Div}(Y)$.

We say that Φ is the contraction of the *extremal face* F and that H is a *good supporting divisor* for Φ.

Remark 1.3.13. A contraction always corresponds to a face in the Kleiman-Mori cone (not necessarily in the negative part) [De01, Section 1.3].

From now on we suppose that X is smooth.

Remark 1.3.14. A good supporting divisor for a Fano-Mori contraction is of the form $H = K_X + rL$, where r is a positive integer and L is an ample line bundle.

Remark 1.3.15. Let $\pi\colon X \longrightarrow Y$ denote a contraction of an extremal face F, supported by $H = \pi^* A$. Let R be an extremal ray in F and $\rho\colon X \longrightarrow W$ the contraction of R. Then π factors through ρ (this is because $\pi^* A \cdot R = 0$).

Definition 1.3.16. Let $\Phi\colon X \longrightarrow Y$ be a Fano-Mori contraction of X and let $E = E(\Phi) = \left\{x \in X \;\middle|\; \dim(\Phi^{-1}\Phi(x)) > 0\right\}$ be the exceptional locus of Φ (if Φ is of fiber type, then $E = X$); let S be an irreducible component of a (non-trivial) fiber F. We define the positive integer l as

$$l := \min\left\{-K_X \cdot C \;\middle|\; C \text{ is a rational curve in } S\right\}.$$

If Φ is an elementary extremal Fano-Mori contraction, i.e. the contraction of an extremal ray R, then l is called the *length* of the ray. A rational curve C such that $-K_X \cdot C = l(R)$ is called a *minimal (extremal) curve.*

Proposition 1.3.17. (Ionescu-Wiśniewski inequality)*[Io86, Theorem 0.4] [Wi91a, Theorem 1.1] The following formula holds:*

$$\dim S + \dim E \geq \dim X + l - 1.$$

In particular, this implies that if Φ is of fiber type then $l \leq \dim X - \dim Y + 1$, and if Φ is birational then $l \leq \dim E - \dim(\Phi(E))$.

Theorem 1.3.18. *[Wi91a, Theorem 2.2] Let X be a manifold of dimension n that admits k Fano-Mori contractions of different extremal rays. If we denote by m_i, $i = 1, \dots, k$ the dimensions of the images of these contractions, then*

$$\sum_{i=1}^{k} (n - m_i) \leq n.$$

Proposition 1.3.19. *[AnBaWi92, Proposition 1.4.1] Let $\Phi\colon X \longrightarrow Y$ be an elementary Fano-Mori contraction.*
If Φ is of fiber type and $\dim Y \leq 2$, then it is equidimensional and Y is smooth.

Proposition 1.3.20. *[AnBaWipr, Proposition 1.1], [AnOc99, Proposition 2.9] Let $\Phi\colon X \longrightarrow Y$ be a contraction of a face such that $\dim X > \dim Y$. Suppose that for every rational curve C in a general fiber of Φ we have $-K_X \cdot C \geq \frac{n+1}{2}$. Then Φ is an elementary contraction except if*

1. *$-K_X \cdot C = \frac{n+2}{2}$ for some rational curve C on X, Y is a point, X is a Fano manifold of pseudoindex $\frac{n+2}{2}$ and $\rho(X) = 2$.*

2. *$-K_X \cdot C = \frac{n+1}{2}$ for some rational curve C on X and $\dim Y \leq 1$.*

1.4 Ample vector bundles

Let $\mathcal{E}$ be a rank r vector bundle on a variety X, and let $S^t(\mathcal{E})$ denote the t-th symmetric power of $\mathcal{E}$ for $t \geq 0$, with the convention $S^0(\mathcal{E}) = \mathcal{O}_X$. Let $\mathcal{S} := \oplus_{t=0}^{\infty} S^t(\mathcal{E})$ be the symmetric algebra of $\mathcal{E}$; then $\mathbb{P}(\mathcal{E}) := \mathrm{Proj}(S)$ is a $\mathbb{P}^d$-bundle on X, with projection $\pi\colon \mathbb{P}(\mathcal{E}) \longrightarrow X$.
There exists an invertible line bundle $\xi_{\mathcal{E}}$ which, over $x \in \mathbb{P}(\mathcal{E})$, is the quotient space of $(\pi^*\mathcal{E})_x$ corresponding to the one dimensional quotient of $\mathcal{E}_{\pi(x)}$ defining x. On $\mathbb{P}(\mathcal{E})$ we have $\xi_F \cong \mathcal{O}_{\mathbb{P}^d}(1)$ for any fiber F of π; the line bundle $\xi_{\mathcal{E}}$ is called the *tautological line bundle* on $\mathbb{P}(\mathcal{E})$. We have $\pi^*\xi_{\mathcal{E}} \cong \mathcal{E}$ and $\pi_*(t\xi_{\mathcal{E}}) = S^t(\mathcal{E})$, for any integer $t \geq 0$. The canonical bundle of $\mathbb{P}(\mathcal{E})$ is given by

$$K_{\mathbb{P}(\mathcal{E})} \equiv \pi^*(K_X + \det \mathcal{E}) - r\xi_{\mathcal{E}}.$$

Definition 1.4.1. We say that a vector bundle is *ample* if the tautological line bundle is ample on $\mathbb{P}(\mathcal{E})$.

We will need the following results:

Lemma 1.4.2. *Let $\mathcal{E}$ be an ample vector bundle of rank r on a complex variety X. For any rational curve $f\colon \mathbb{P}^1 \longrightarrow C \subset X$ we have*

$$\det \mathcal{E} \cdot C \geq r.$$

Moreover, if C is smooth and the equality holds, then $\mathcal{E}_C = \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus r}$.

Proof. Since every vector bundle on $\mathbb{P}^1$ splits as a direct sum of line bundles and $\mathcal{E}$ is ample, we have that

$$f^*\mathcal{E} = \oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i), \quad \text{with all } a_i \geq 1.$$

Therefore $\det \mathcal{E} \cdot C = \sum_{i=1}^r a_i \geq r$. *q.e.d.*

Lemma 1.4.3. *Let X be a complex projective variety of dimension n, $\mathcal{E}$ an ample vector bundle on X of rank r, $s \in \Gamma(\mathcal{E})$. Then, if $Z := (s)_0$ denotes the zero set of s,*

$$\dim Z \geq n - r.$$

Proof. See [Ful84, Example 12.1.3] *q.e.d.*

1.4.4. Let $\mathcal{E}$ be an ample vector bundle of rank r on a complex projective manifold X of dimension n such that there exists a section $s \in \Gamma(\mathcal{E})$ whose zero locus $Z = (s)_0$ is a smooth submanifold of X of the expected dimension $n - r \geq 2$.

Remark 1.4.5. With these assumptions, the restriction of $\mathcal{E}$ to Z is the normal bundle $N_X Z$.

Proposition 1.4.6. *Let X, $\mathcal{E}$ and Z be as in 1.4.4. Let Y be a subvariety of X of dimension $\geq r$. Then* $\dim Z \cap Y \geq \dim Y - r$.

Proof. We consider the restrictions $\mathcal{E}_Y$ of $\mathcal{E}$ to Y and s_Y of s to $\Gamma(Y, \mathcal{E}_Y)$. If $S := (s_Y)_0$ denotes the zero set of s_Y, by 1.4.3 applied to Y and s_Y, we find

$$\dim Z \cap Y = \dim S \geq \dim Y - r.$$

q.e.d.

In order to investigate the relation between the cones $\overline{NE}(X)$ and $\overline{NE}(Z)$, the following Lefschetz type theorem essentially due to A.J. Sommese ([So78]) will be useful.

Theorem 1.4.7. *[LaMa96, Theorem 1.1] Let $X, \mathcal{E}$ and Z be as in* (1.4.4) *and let $i\colon Z \hookrightarrow X$ be the embedding. Then*

1. *$H^i\colon H^i(X, \mathbb{Z}) \longrightarrow H^i(Z, \mathbb{Z})$ is an isomorphism for $i \leq \dim Z - 1$.*
2. *H^i is injective and its cokernel is torsion free for $i = \dim Z$.*
3. Pic: $\mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(Z)$ *is an isomorphism for* $\dim Z \geq 3$.
4. Pic *is injective and its cokernel is torsion free for* $\dim Z = 2$.
5. *$\rho(X) = \rho(Z)$ for* $\dim Z \geq 3$.

1.5 Fano manifolds

In this section we recall some results on Fano manifolds.

Definition 1.5.1. A projective manifold X is called a *Fano manifold* if its anticanonical divisor $-K_X$ is an ample Cartier divisor.
We define the *index* of a Fano manifold X as the largest integer dividing $-K_X$ in the Picard group $\mathrm{Pic}(X)$, i.e.

$$r_X := \mathrm{index}(X) := \max \left\{ k \in \mathbb{Z} \;\middle|\; -K_X = kH \text{ for some ample divisor } H \right\}$$

and the *pseudoindex* as

$$i_X := \min \left\{ -K_X \cdot C \;\middle|\; C \text{ is a rational curve on } X \right\}.$$

Remark 1.5.2. If X is a Fano manifold, then $\mathrm{Pic}(X)$ is torsion free.

Theorem 1.5.3. *[KoOc73] Let X be a Fano manifold of dimension n with rational singularities. Let L be an ample Cartier divisor on X such that $-K_X = \mathrm{index}(X)L$. Then*

1. $\mathrm{index}(X) \leq n+1$;
2. $\mathrm{index}(X) = n+1$ *if and only if* $(X, L) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$;
3. $\mathrm{index}(X) = n$ *if and only if* $(X, L) = (\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$.

Definition 1.5.4. Let X be a smooth projective variety of dimension n and let L be an ample line bundle on it. The pair (X, L) is called a *del Pezzo manifold* (resp. a *Mukai manifold*) if $-K_X = (n-1)L$ (resp. $-K_X = (n-2)L$).

We now give two theorems on Fano manifolds proved by J.A. Wiśniewski, since they will be used throughout all chapter 3.

Theorem 1.5.5. *[Wi90a, Theorem A] Let X be a Fano manifold of pseudoindex i_X. If $i_X > \frac{1}{2}\dim X + 1$, then $\mathrm{Pic}(X) = \mathbb{Z}$.*

Theorem 1.5.6. *[Wi90a, Theorem B] Let X be a Fano manifold of index r_X. If $\dim X \leq 2r_X - 2$, then $\mathrm{Pic}(X) = \mathbb{Z}$ unless $X \cong \mathbb{P}^{r_X-1} \times \mathbb{P}^{r_X-1}$.*

Definition 1.5.7. Let X be a smooth projective variety of dimension n and let L be an ample line bundle on it.
We say that the pair (X, L) is a *scroll* (resp. a *(hyper)quadric fibration*, resp. a *del Pezzo fibration*, resp. a *Mukai fibration*) over a normal variety Y of dimension m if there exists a surjective morphism with connected fibers $\Phi\colon X \longrightarrow Y$ such that $(K_X+(n-m+1)L) \sim \Phi^*\mathcal{L}$ (resp. $(K_X+(n-m)L) \sim \Phi^*\mathcal{L}$, resp. $(K_X+(n-m-1)L) \sim \Phi^*\mathcal{L}$, resp. $(K_X + (n-m-2)L) \sim \Phi^*\mathcal{L}$) for some ample line bundle $\mathcal{L}$ on Y.
We say that X is a $\mathbb{P}$-*bundle* (resp. a *quadric bundle*) over a projective variety Y of dimension m if there exists a surjective morphism $\Phi\colon X \longrightarrow Y$ such that every fiber is isomorphic to $\mathbb{P}^{n-m}$ (resp. to a hyperquadric $\mathbb{Q}^{n-m} \subset \mathbb{P}^{n-m+1}$) and if there exists a vector bundle $\mathcal{E}$ of rank $n-m+1$ (resp. $n-m+2$) on Y such that $X \cong \mathbb{P}(\mathcal{E})$ (resp. there exists an embedding of X over Y as a divisor of $\mathbb{P}(\mathcal{E})$ of relative degree 2).

For further details on the definitions of these special varieties we refer to [BeSo95].

Remark 1.5.8. A scroll is a Fano-Mori contraction of fiber type such that the inequality in 1.3.17 is actually an equality, i.e. $l = \dim X - \dim Y + 1$, and moreover if C is a rational curve such that $-K_X \cdot C = l$ then there exists an ample line bundle L such that $L \cdot C = 1$, i.e. C is a line with respect to L.
The contrary is almost true, in the sense that if Φ is a Fano-Mori contraction with the above properties then it factors through a scroll, i.e. the face contracted by Φ contains a sub-face whose contraction is a scroll.
Similarly a (hyper)quadric fibration is a Fano-Mori contraction of fiber type such that $l = \dim X - \dim Y$, and moreover if C is a rational curve such that $-K_X \cdot C = l$ then there exists an ample line bundle L such that $L \cdot C = 1$, i.e. C is a line with respect to L.

In the following we will denote by $\mathbb{Q}^n$ the smooth hyperquadric in the projective space $\mathbb{P}^{n+1}$.

Chapter 2

Manifolds polarized by vector bundles

Let X be a smooth complex projective variety of dimension $n \geq 3$ and let $\mathcal{E}$ be an ample vector bundle of rank r on X.

In this chapter, which contains the results of [AnNopr], we want to give suitable assumptions under which it is possible to classify polarized varieties $(X, \mathcal{E})$.

A famous theorem of S. Mori ([Mo79]) says that if $\mathcal{E}$ is the tangent bundle of X, then X is the projective space; this result has been generalized by M. Andreatta and J.A. Wiśniewski with the assumption that $\mathcal{E}$ is just a subsheaf of the tangent bundle of X ([AnWi01]).

For this purpose, in the spirit of Mori theory we assume that X is not minimal, i.e. that K_X is not nef. Then we can define the nef value $\tau := \tau(X, \det \mathcal{E}) := \min\{t \in \mathbb{R} : K_X + t \det \mathcal{E} \text{ is nef}\}$ of the polarized variety $(X, \det \mathcal{E})$, which is a positive (since K_X is not nef) rational number (by the Kawamata's Rationality Theorem (1.3.4)).

Moreover the divisor $K_X + \tau \det \mathcal{E}$ defines a face $F(\mathcal{E}) := \{C \in \overline{NE}(X) : (K_X + \tau \det \mathcal{E}) \cdot C = 0\}$ in the polyhedral part, $\overline{NE}_{K_X<0}(X)$, of the Kleiman-Mori cone. This face is generated by a finite number of extremal rays $R_i = \mathbb{R}[C_i]$, where C_i is a rational curve. For any extremal ray $R \subset \overline{NE}_{K_X<0}(X)$, we define the length of the ray, i.e. the integer $l(R) :=\min \{-K_X \cdot C \mid [C] \in R\}$, which satisfies $l(R) \leq n+1$, by a theorem of Mori (1.3.1).

To any ray R, we can associate a map $\varphi\colon X \to Y$ into a normal projective variety with connected fibers (1.3.12) such that $-K_X$ is φ-ample, therefore φ is a Fano-Mori contraction and it contracts all curves whose numerical class is in R.

Working with extremal rays in $F(\mathcal{E})$, we give a classification of pairs $(X, \mathcal{E})$ under suitable assumptions on the nef value $\tau := \tau(X, \det \mathcal{E})$ of the polarized variety $(X, \det \mathcal{E})$ for all $\tau \geq \frac{n-2}{r}$.

We start this chapter recalling some known facts on families of rational curves (for which we refer to [Ko96]) and giving a bound on the nef value τ of the polarized variety $(X, \det \mathcal{E})$ and a condition under which the vector bundle $\mathcal{E}$ splits as a direct sum $\mathcal{E} = L^{\oplus r}$, for some ample line bundle L on X (2.1.10).

2.1 Families of rational curves

Assume that X is a normal projective variety and let $\mathrm{Hom}(\mathbb{P}^1, X)$ be the scheme parametrizing morphism $f\colon \mathbb{P}^1 \longrightarrow X$.
We denote by $F\colon \mathrm{Hom}(\mathbb{P}^1, X) \times \mathbb{P}^1 \longrightarrow X$ the evaluation morphism $F(f,p) = f(p)$.

Definition 2.1.1. We call a *family of rational curves* any closed irreducible component $V \subset \mathrm{Hom}(\mathbb{P}^1, X)$.

For any family of morphisms $V \subset \mathrm{Hom}(\mathbb{P}^1, X)$, we denote by F_V the restriction of F to V and by $\mathrm{Locus}(V)$ the image of F_V.

Definition 2.1.2. We say that V is *unsplit* if the image of V in $\mathrm{Chow}(X)$ is proper.

Remark 2.1.3. Let $R = \mathbb{R}^+[C]$ be an extremal ray of a smooth variety X. Assume that the anticanonical degree of $[C]$ is minimal in R.
Then the irreducible component of $\mathrm{Hom}(\mathbb{P}^1, X)$ containing C is unsplit.
Indeed, if C is reducible as a cycle, then its components, being R an extremal ray, belong to R. But this is a contradiction with the minimality of C with respect to the intersection with the anticanonical bundle.

Definition 2.1.4. Let X be a projective variety.
X is *rationally chain connected* if and only if for any very general closed points $x_1, x_2 \in X$, there exists a connected curve $C \subset X$ which contains x_1 and x_2 such that every irreducible component of C is rational.
X is *rationally connected* if and only if for any very general closed points $x_1, x_2 \in X$, there exists an irreducible rational curve $C \subset X$ which contains x_1 and x_2.

Remark 2.1.5. If X is rationally connected, then X is rationally chain connected.

Remark 2.1.6. We note that the previous definition can be stated for varieties defined over any algebraically closed field (see [Ko96]).

An unsplit family V defines a relation of *rational connectedness with respect to* V, rc-V relation: $x_1, x_2 \in X$ are in the rc-V relation if there exists a chain of rational curves parametrized by morphisms from V which joins x_1 and x_2. The rc-V relation is an equivalence relation and its equivalence classes can be parametrized by an algebraic set (see [Ko96]).

Proposition 2.1.7. *[AnWi01, Proposition 1.1] Let X be a projective manifold and $V \subset \mathrm{Hom}(\mathbb{P}^1, X)$ an unsplit family of rational curves whose locus is the whole X. Then X is rc-V if and only if $\rho(X) = 1$.*

Note that rationally connectedness and $\rho(X) = 1$ imply that X is Fano.

Proposition 2.1.8. *[AnWi01, Proposition 1.2] Let X be a projective manifold, $\mathcal{E}$ a vector bundle on X of rank r and $V \subset \mathrm{Hom}(\mathbb{P}^1, X)$ an unsplit family of rational curves whose locus is the whole X. Suppose moreover that there exists an integer a such that for any $f \in V$ we have $f^*\mathcal{E} = \mathcal{O}(a)^{\oplus r}$. Then there exists a (uniquely defined) line bundle L over X such that $\deg f^*L = a$ and $\mathcal{E} \cong L^{\oplus r}$.*

Now and in the rest of this chapter we will work under the following assumptions.

2.1.9. Let $(X, \mathcal{E})$ be a polarized variety, given by a complex projective manifold X of dimension $n \geq 3$ and an ample vector bundle $\mathcal{E}$ of rank r on X.
Assume that X is not minimal, i.e. that K_X is not nef.
Let τ be the nef value $\tau := \tau(X, \det \mathcal{E}) := \min\{t \in \mathbb{R} \,|\, K_X + t \det \mathcal{E} \text{ is nef}\}$ of the pair $(X, \det \mathcal{E})$.

We give first a bound on the nef value τ in terms of the rank, r, of $\mathcal{E}$ and of the length, $l(R)$, of the extremal ray R. Moreover we give a condition under which the vector bundle splits as a direct sum $\mathcal{E} = L^{\oplus r}$ for some ample line bundle L on X.

Theorem 2.1.10. *[AnNopr, Theorem 1.3] Let X be a smooth complex projective variety of dimension n and let R be any extremal ray in the face $F(\mathcal{E}) := \{Z \in N_1(X) \,|\, (K_X + \tau \det \mathcal{E}) \cdot Z = 0\}$, where τ is the nef value of the pair $(X, \det \mathcal{E})$. Assume that K_X is not nef. Let $C \subset X$ be any rational curve such that $l(R) = -K_X \cdot C$ and $[C] \in R$. Then*

$$\tau := \tau(X, \det \mathcal{E}) \leq \frac{l(R)}{r} \left(\leq \frac{n+1}{r} \right).$$

Moreover

1. *equality holds if and only if* $\det \mathcal{E} \cdot C = r$*, and if* V *is a family of rational curves which contains* $f \colon \mathbb{P}^1 \to C \subset X$ *then it is unsplit.*

2. *If equality holds and* X *is rationally chain connected with respect to* V *, which is equivalent to assume* $\rho(X) = 1$*, then there exists a (uniquely defined) line bundle* L *over* X *such that* $\deg f^*L = 1$ *and* $\mathcal{E} \cong L^{\oplus r}$*.*

Proof. Assume by contradiction that $\tau(X, \mathcal{E}) > \frac{l(R)}{r}$. Then, if C is a minimal rational curve in R, we have

$$0 = (K_X + \tau \det \mathcal{E}) \cdot C > K_X \cdot C + \frac{l(R)}{r} \det \mathcal{E} \cdot C = K_X \cdot C \left(1 - \frac{\det \mathcal{E} \cdot C}{r}\right).$$

This implies that $\det \mathcal{E} \cdot C < r$, which is a contradiction since $\mathcal{E}$ is ample.
In the same way one proves that equality holds if and only if $\det \mathcal{E} \cdot C = r$.
The rest of the theorem follows from 2.1.7 and 2.1.8. *q.e.d.*

Remark 2.1.11. The assumption that the base field is the complex number is used in the proof of 2). It would be nice to have a proof of it over an arbitrary algebraically closed field.
Note also that part 2) will be used to reduce the general case to the case $r = 1$.

2.2 Pairs $(X, \mathcal{E})$ with $\frac{n+1}{r} \leq \tau(X, \det \mathcal{E})$

In view of 2.1.10, we can confine to consider case $\tau = \frac{n+1}{r}$.

Proposition 2.2.1. *Let* X*,* $\mathcal{E}$ *and* τ *be as in 2.1.9.*
If $\tau = \frac{n+1}{r}$*, then* $(X, \mathcal{E}) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus r})$*.*

Proof. Let R be any extremal ray in the face $F(\mathcal{E}) := \{Z \in N_1(X) \mid (K_X + \tau \det \mathcal{E}) \cdot Z = 0\}$.
By 2.1.10 we have that $l(R) = n + 1$. Then we have, by [ChMiSB01], that $X = \mathbb{P}^n$ and, by 2.1.10, that $\mathcal{E} = L^{\oplus r}$ for a line bundle L over X. Therefore $\tau(X, L) = n + 1$ and we reduce our proposition to the known case $r = 1$. *q.e.d.*

We note that the proposition is an improvement of [Oh99, Proposition 0.1], where M. Ohno assumes $r \leq n$ and it was first proved by M. Andreatta in [And03, Theorem 4].

2.3 Pairs $(X,\mathcal{E})$ with $\frac{n}{r}\leq\tau(X,\det\mathcal{E})<\frac{n+1}{r}$

Proposition 2.3.1. *Let X, $\mathcal{E}$ and τ be as in 2.1.9.*
Assume $\frac{n}{r}\leq\tau<\frac{n+1}{r}$ and let $a:=\det\mathcal{E}\cdot C-r$.
Then one of the following holds:

1) *$X=\mathbb{P}^n$, $a\geq 1$ and $an\leq r$. If $r\leq n$ then $\mathcal{E}$ is either $T_{\mathbb{P}^n}$ or $\mathcal{O}_{\mathbb{P}^n}(2)\oplus\mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(r-1)}$.*

2) *$X=\mathbb{Q}^n$ and $\mathcal{E}=\mathcal{O}_{\mathbb{Q}^n}(1)^{\oplus r}$.*

3) *X is the projectivization of a rank n vector bundle on a smooth curve Y, $\pi\colon\mathbb{P}(\mathcal{F})\to Y$, and $\mathcal{E}_{|F}=\mathcal{O}_{\mathbb{P}^{n-1}}(1)^{\oplus r}$ for every fiber F of π.*

Proof. Let R be any extremal ray in the face $F(\mathcal{E}):=\{Z\in N_1(X)\mid(K_X+\tau\det\mathcal{E})\cdot Z=0\}$.
By 2.1.10 we have that $l(R)\geq n$.
If $l(R)=n+1$ then by [ChMiSB01] we have that $X=\mathbb{P}^n$. Moreover $\frac{n}{r}\leq\tau=\frac{-K_X\cdot C}{\det\mathcal{E}\cdot C}=\frac{n+1}{r+a}<\frac{n+1}{r}$ gives the bounds on a.
If $r\leq n$, then $a=1, r=n$ and thus $\tau=1$ and the theorem follows from [Pe91].
If $l(R)=n$ and $\rho(X)=1$, by 2.1.10 we have that $\mathcal{E}=L^{\oplus r}$ for a line bundle L over X such that $\deg f^*L=1$, with $f\colon\mathbb{P}^1\longrightarrow C\subset X$ minimal curve. Therefore $\tau(X,L)=n$ and we reduce our proposition to the known case $r=1$. This gives the case 2) of the proposition.
Let $l(R)=n$ and $\rho(X)>1$; by 1.3.17 and 1.3.19 the map $\varphi_R\colon X\to Y$ is equidimensional and onto a smooth curve. If F is a general fiber the pair $(F,\mathcal{E}_{|F})$ is $(\mathbb{P}^{n-1},\mathcal{O}_{\mathbb{P}^{n-1}}(1)^{\oplus r})$ by 2.2.1. Using the same argument as in [Fu90b, Section 3.3] we see that this is true for every fiber F; therefore we are in case 3) of the proposition.
q.e.d.

We note that the proposition is an improvement of [Oh99, Proposition 0.2 and Theorem 0.3], where M. Ohno assumes $r\leq n$.

2.4 Pairs $(X,\mathcal{E})$ with $\frac{n-1}{r}\leq\tau(X,\det\mathcal{E})<\frac{n}{r}$

Proposition 2.4.1. *Let X, $\mathcal{E}$ and τ be as in 2.1.9.*
Assume $\frac{n-1}{r}\leq\tau<\frac{n}{r}$ and let $a:=\det\mathcal{E}\cdot C-r$.
Then one of the following holds:

a) $\rho(X)=1$ and

1) *$X = \mathbb{P}^n$, a is a positive integer and $\frac{n-1}{2}a \le r < na$. In particular if $r \le n-1$ (for instance if $\tau \ge 1$) then either $a = 1, r \ge \frac{n-1}{2}$ and $\mathcal{E} = \mathcal{O}_{\mathbb{P}^n}(2) \oplus \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(r-1)}$ or $a = 2, r = n-1, \tau = 1$ and the possible $\mathcal{E}$ are described in [PeSzWi92].*

2) *X is a Fano manifold, $-K_X \cdot C = n$ for every minimal (rational) curve, $a \ge 1$ and $a(n-1) \le r$, $\tau \le \frac{1}{a}$. In particular if $r \le n$ then $X = \mathbb{Q}^n$ and $\mathcal{E}$ is uniform with splitting type $(2, 1, \ldots, 1)$ (and, for $r = n-1$, it is described in [PeSzWi92]).*

3) *there exists an ample line bundle L over X such that $-K_X = (n-1)L$ (i.e. (X, L) is a del Pezzo manifold) and $\mathcal{E} \cong L^{\oplus r}$.*

b) $\rho(X) > 1$ and

4) *X is the projectivization of a rank n vector bundle on a smooth curve Y, $\pi\colon \mathbb{P}(\mathcal{F}) \to Y$, $a \ge 1$ and $a(n-1) \le r$. If $r \le (n-1)$ (for instance if $\tau \ge 1$) then for every fiber F of π the pair $(F, \mathcal{E}_{|F})$ is as in 1) of 2.3.1.*

5) *X is a section of a divisor of relative degree 2 in a $(n+1)$-dimensional projectivization of a vector bundle over a smooth curve Y and for every smooth fiber F the pair $(F, \mathcal{E}_{|F})$ is as in 2) of 2.3.1.*

6) *X is a $\mathbb{P}^{n-2}$-bundle over a smooth surface S, locally trivial in the complex topology, and $\mathcal{E}_{|F} = \mathcal{O}_{\mathbb{P}^{n-2}}(1)^{\oplus r}$ for every fiber F of π (see also the following remark).*

Or

7) *There exist a smooth variety X' and a morphism $\varphi\colon X \to X'$ expressing X as blow-up of X' at a finite set of points B and an ample vector bundle $\mathcal{E}'$ on X' such that $\mathcal{E} \otimes ([\varphi^{-1}(B)]) = \varphi^*\mathcal{E}'$ and $K_{X'} + \tau \det \mathcal{E}'$ is ample.*
Moreover $\mathcal{E}_{|E} = \mathcal{O}_{\mathbb{P}^{n-1}}(1)^{\oplus r}$, where E is any irreducible component of the exceptional locus of φ.
The pair $(X', \mathcal{E}')$ is called the first reduction of *$(X, \mathcal{E})$.*

Proof. Let R be any extremal ray in the face $F(\mathcal{E}) := \{Z \in N_1(X) \mid (K_X + \tau \det \mathcal{E}) \cdot Z = 0\}$.
By 2.1.10 we have that $l(R) \ge n-1$.

If $l(R) = n+1$ then by [ChMiSB01] we have that $X = \mathbb{P}^n$. If $a = 0$ we can apply 2.1.10 and $\mathcal{E} = \oplus^r \mathcal{O}_{\mathbb{P}^n}(1)$, which is a contradiction.
Since $r + a = \det \mathcal{E} \cdot C = \frac{l(R)}{\tau} = \frac{n+1}{\tau}$, we have that $\frac{n-1}{2} a \le r < na$. In particular if $r \le n-1$ then $a = 1, 2$. If $a = 1$ then $r \ge \frac{n-1}{2}$ and $\mathcal{E}$ is uniform with splitting type $(2, 1, \ldots, 1)$, therefore $\mathcal{E} = \mathcal{O}_{\mathbb{P}^n}(2) \oplus \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(r-1)}$. If $a = 2$ then $r = n-1, \tau = 1$ and all possible $\mathcal{E}$ are described in [PeSzWi92].
If $l(R) = n$ and $\rho(X) = 1$, we can assume again by 2.1.10 that $a \ge 1$. Moreover $r + a = \frac{n}{\tau} \le \frac{nr}{n-1}$ implies $a(n-1) \le r$, $\tau \le \frac{1}{a}$. If $r \le n$ then $a = 1$ and therefore $-K_X \cdot C = n$ and $\det \mathcal{E} \cdot C = n$ or $\det \mathcal{E} \cdot C = n+1$. In the first case $\tau = 1$, $r = n-1$ and we conclude using [PeSzWi92] and [Ocpr2]. In the second, since n and $n+1$ are relatively prime, we can find an ample line bundle H such that $H \cdot C = 1$ and therefore such that $\tau(X, H) = n$. We can now apply the known case $r = 1$.
If $l(R) = n-1$ and $\rho(X) = 1$, then we are in the assumption of 2.1.10. Then $\mathcal{E} \cong L^{\oplus r}$ for a line bundle L over X such that $\deg f^*L = 1$, with $f\colon \mathbb{P}^1 \longrightarrow C \subset X$ minimal curve, and we are in the case 3) of the proposition.
If $l(R) = n$ and $\rho(X) > 1$, then, as in the proof of 2.3.1, it is straightforward to see that the map X is the projectivization of a rank n vector bundle on a smooth curve Y, $a \ge 1$ and $a(n-1) \le r$. The rest of point 4) follows from 2.3.1 applied to the pair $(F, \mathcal{E}_{|F})$.
Let $l(R) = n-1$ and $\rho(X) > 1$; if $\varphi_R\colon X \to Y$ is of fiber type then by 1.3.17 and 1.3.19 it is equidimensional and onto either a smooth curve or a smooth surface. If F is a general fiber the pair $(F, \mathcal{E}_{|F})$ is $(\mathbb{Q}^{n-1}, \mathcal{O}_{\mathbb{Q}^{n-1}}(1)^{\oplus r})$ by 2.3.1 in the first case, and $(\mathbb{P}^{n-2}, \mathcal{O}_{\mathbb{P}^{n-2}}(1)^{\oplus r})$ by 2.2.1 in the second. In the first case, using the same arguments as in [Fu90b, Section 3.3], we see that every fiber F is a possibly singular hyperquadric and then that X is a section of a divisor of relative degree 2 in a $(n+1)$-dimensional projectivization of a vector bundle over a smooth curve Y. Also in the second case, using this time the argument in [AnBaWi92, 2.2], one can see that this is true for every fiber F and then that $\varphi_R\colon X \to Y$ is a $\mathbb{P}^{n-2}$-bundle, locally trivial in the complex topology.
We are therefore left with the case $l(R) = n-1$, $\rho(X) > 1$ and $\varphi_R\colon X \to Y$ birational (if the last assumption holds it is usually said that the ray R is not nef). Then, by [AnOc02b, Theorem 1.1], Y is smooth and φ_R is the blow-up of Y at a point. Moreover, if E denotes the exceptional locus of φ_R, by adjunction $\det \mathcal{E}_{|E} = \mathcal{O}(r)$, therefore $(E, \mathcal{E}_{|E}) = (\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1)^{\oplus r})$.
Also the same proof as the one of [AnOc02b, Lemma 4.2] proves that if a ray R of the face $F(\mathcal{E})$ with $\tau \ge \frac{n-1}{r}$ (which implies $\tau = \frac{n-1}{r}$, by 2.1.10) is non nef then all rays in the face are non nef with the only exception given by the blow-up of $\mathbb{P}^3$ in one point, $\pi\colon Bl_x\mathbb{P}^3 \to \mathbb{P}^3$, and $\mathcal{E} = (\pi^*(\mathcal{O}_{\mathbb{P}^3}(2)) - [\pi^{-1}(x)])^{\oplus r}$. But this last case is already contained in 6).
Also the same proof as the one of [AnOc02b, Theorem 4.1] proves that there exist a

smooth variety X' and a morphism $\varphi\colon X \longrightarrow X'$ expressing X as blow-up of X' at a finite set of points B. Now, by [AnOc02a, Lemma 2.9] there exists a rank r vector bundle $\mathcal{E}'$ on X' such that $\mathcal{E} \otimes ([\varphi^{-1}(B)]) = \varphi^*\mathcal{E}'$. Applying now [AnOc02a, Lemma 2.10], we find that $\mathcal{E}'$ is ample.
Moreover $\varphi^*(K_{X'} + \tau \det \mathcal{E}') = K_X - \sum(n-1)E_i + \tau \det \mathcal{E} + \tau r \sum E_i = K_X + \tau \det \mathcal{E}$ and we conclude, since $K_X + \tau \det \mathcal{E} = \varphi^*(A)$, where A is an ample divisor on X', and the map $\varphi^*\colon \mathrm{Pic}(X') \longrightarrow \mathrm{Pic}(X)$ is injective.
All this leads to the last case of the proposition. *q.e.d.*

Remark 2.4.2. In part 2) of the proposition X should be the hyperquadric also if $r \geq n$.

Remark 2.4.3. In part 6) of the proposition X is not necessarily a projectivization of a vector bundle, as it is shown in [AnBaWi92, Example 2.3]. The example in particular says also that the assumption $\rho(X) = 1$ is necessary in part 2) of 2.1.10. Moreover note that part 6) contains also the blow-up of $\mathbb{P}^3$ in one point, $\pi\colon Bl_x\mathbb{P}^3 \to \mathbb{P}^3$, and $\mathcal{E} = (\pi^*(\mathcal{O}_{\mathbb{P}^3}(2)) - [\pi^{-1}(x)])^{\oplus r}$.

We note that the proposition is an improvement of [Oh99, Theorem 0.4 and 0.5], where M. Ohno assumes $r \leq n-1$.

2.5 Pairs $(X, \mathcal{E})$ with $\frac{n-2}{r} \leq \tau(X, \det \mathcal{E}) < \frac{n-1}{r}$

Proposition 2.5.1. *Let X, $\mathcal{E}$ and τ be as in 2.1.9.*
Assume $\frac{n-2}{r} \leq \tau < \frac{n-1}{r}$ and let $a := \det \mathcal{E} \cdot C - r$.
Then one of the following holds:

a) $\rho(X) = 1$ and

1) *$X = \mathbb{P}^n$, a is a positive integer and $\frac{n-2}{3}a \leq r < \frac{n-1}{2}a$. In particular, if $r \leq n-2$ (for instance if $\tau \geq 1$), then $a = 1, 2, 3$ and $\mathcal{E}$ is a decomposable bundle.*

2) *X is a Fano manifold, $-K_X \cdot C = n$ for every minimal rational curve, $a \geq 1$ and $\frac{n-2}{2}a \leq r < (n-1)a$.*

3) *X is a Fano manifold, $-K_X \cdot C = n-1$ for every minimal rational curve, $a \geq 1$ and $(n-2)a \leq r$. If $r \leq n-2$ (for instance if $\tau \geq 1$) then $K_X + \det \mathcal{E} = 0$.*

4) *there exists an ample line bundle* L *over* X *such that* $-K_X = (n-2)L$ *(i.e.* (X, L) *is a Mukai manifold) and* $\mathcal{E} \cong L^{\oplus r}$.

b) $\rho(X) > 1$ *and*

5) X *is the projectivization of a rank* n *vector bundle over a smooth curve* Y, $\pi\colon \mathbb{P}(\mathcal{F}) \longrightarrow Y$, $\frac{n-2}{2}a \le r < a(n-1)$ *and for every fiber* F *the pair* $(F, \mathcal{E}_{|F})$ *is as in 1) of 2.4.1. In particular, if* $r \le n-2$ *(for instance if* $\tau \ge 1$*), then either* $r \ge \frac{n-2}{2}$ *and* $\mathcal{E}_{|F} = \mathcal{O}_{\mathbb{P}^{n-1}}(2) \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(1)^{\oplus(r-1)}$, *or* $r = n-2$, $\tau = 1$ *and the possible* $\mathcal{E}_{|F}$ *are described in [PeSzWi92].*

6) X *is a Fano fibration over a smooth curve* Y *and* $r \ge (n-2)a$. *In particular, if* $r \le n-2$ *(for instance if* $\tau \ge 1$*), then* X *is a section of a divisor of relative degree* 2 *in a* $(n+1)$*-dimensional projectivization of a vector bundle over a smooth curve* Y *and for every fiber* F $\mathcal{E}_{|F}$ *is uniform with splitting type* $(2, 1, \ldots, 1)$ *(and it is described in [PeSzWi92]).*

7) X *is a fibration over a smooth curve* Y*; for the general fiber* F, $-K_F = (n-2)L$, $\mathcal{E}_{|F} = L^{\oplus r}$ *and* (F, L) *is a del Pezzo manifold.*

8) X *is a fibration over a smooth surface* S *and* $r \ge (n-2)a$ *and, for the general fiber* F, $F = \mathbb{P}^{n-2}$. *In particular if* $r \le n-2$ *(for instance if* $\tau \ge 1$*) then* X *is a* $\mathbb{P}^{n-2}$*-bundle and for every fiber* F *either* $\mathcal{E}_{|F} = T_{\mathbb{P}^{n-2}}$ *or* $\mathcal{E}_{|F} = \mathcal{O}_{\mathbb{P}^{n-2}}(2) \oplus \mathcal{O}_{\mathbb{P}^{n-2}}(1)^{\oplus(r-1)}$.

9) X *is a fibration over a smooth surface* S *and, for the general fiber* F, *the pair* $(F, \mathcal{E}_{|F}) = (\mathbb{Q}^{n-2}, \mathcal{O}_{\mathbb{Q}^{n-2}}(1)^{\oplus r})$.

10) X *is a fibration over a threefold* T *with at most isolated rational and Gorenstein singularities and for all fibers* F *over a smooth point the pair* $(F, \mathcal{E}_{|F}) = (\mathbb{P}^{n-3}, \mathcal{O}_{\mathbb{P}^{n-3}}(1)^{\oplus r})$.

Or

11) f_R *is the blow up of a smooth variety either in a point or along a smooth curve with exceptional locus* E. *In the first case if* $r \le n-2$ *(for instance if* $\tau \ge 1$*) then* $r = n-2$ *and* $\mathcal{E}_{|E} = \mathcal{O}_{\mathbb{P}^{n-1}}(2) \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(1)^{\oplus(r-1)}$. *In the second case,* $(F, \mathcal{E}_{|F}) = (\mathbb{P}^{n-2}, \mathcal{O}_{\mathbb{P}^{n-2}}(1)^{\oplus r})$ *for all fibers* $F \subset E$.

12) f_R is a divisorial contraction whose exceptional locus, E, satisfies one of the following:

i) $(E, E_E; \mathcal{E}_{|E}) = (\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(-2); \mathcal{O}_{\mathbb{P}^{n-1}}(1)^{\oplus r})$;

ii) $(E, E_E; \mathcal{E}_{|E}) = (\mathbb{Q}^{n-1}, \mathcal{O}_{\mathbb{Q}^{n-1}}(-1); \mathcal{O}_{\mathbb{Q}^{n-1}}(1)^{\oplus r})$.

Proof. Let R be any extremal ray in the face $F(\mathcal{E}) := \{Z \in N_1(X) \mid (K_X + \tau \det \mathcal{E}) \cdot Z = 0\}$.
By 2.1.10 we have that $l(R) \geq n-2$.
If $l(R) = n+1$, then by [ChMiSB01] we have that $X = \mathbb{P}^n$ and the rest is straightforward.
Assume first that $\rho(X) = 1$.
If $l(R) = n$, then $\tau = \frac{n}{r+a}$ and this implies $\frac{n-2}{2}a \leq r < a(n-1)$.
If $l(R) = n-1$, then $\tau = \frac{n-1}{r+a}$ and this implies $(n-2)a \leq r$ and $a > 0$. If $r \leq (n-2)$, then $a = 1$, $r = n-2$, $\tau = 1$ and therefore $K_X + \det \mathcal{E} = 0$.
If $l(R) = n-2$, then we are in the assumption of 2.1.10. In particular $\mathcal{E} \cong L^{\oplus r}$ for a line bundle L over X such that $\deg f^* L = 1$, with $f \colon \mathbb{P}^1 \longrightarrow C \subset X$ minimal curve, and we are in the case 4) of the proposition.
Assume then that $\rho(X) > 1$ and let $\varphi := \varphi_R \colon X \to Y$ be the map associated to the ray R. Since $\rho(X) > 1$ then $\dim Y > 0$.
If $l(R) = n$, then, as in the proof of 2.3.1, it is straightforward to see that X is the projectivization of a rank n vector bundle over a smooth curve and that the pair $(F, \mathcal{E}_{|F})$ is as in 1) of 2.4.1.
Let $l(R) = n-1$ and assume φ is of fiber type; then by 1.3.17 and 1.3.19 it is equidimensional and onto either a smooth curve or a smooth surface. In the first case, if F is a general fiber, the pair $(F, \mathcal{E}_{|F})$ is as in 2) of 2.4.1. In particular if $r \leq n-2$ then $r = n-2$. Since $F = \mathbb{Q}^{n-1}$ and $\mathcal{E}_{|F}$ is uniform with splitting type $(2, 1, \ldots, 1)$ (and it is described in [PeSzWi92]), using the same arguments as in [Fu90b, Section 3.3], we see that every fiber is a possibly singular hyperquadric and that X is a section of a divisor of relative degree 2 in a $(n+1)$-dimensional projectivization of a vector bundle over a smooth curve Y.
In the second case if F is a general fiber then $F = \mathbb{P}^{n-2}$ and the pair $(F, \mathcal{E}_{|F})$ is as in 1) of 2.3.1. In particular if $r \leq n-2$ then $r = n-2$ and $\mathcal{E}_{|F} = T_{\mathbb{P}^{n-2}}$ or $\mathcal{O}_{\mathbb{P}^{n-2}}(2) \oplus \mathcal{O}_{\mathbb{P}^{n-2}}(1)^{\oplus(r-1)}$. Using this time the argument in [AnBaWi92, 2.2], one can see that this is true for every fiber F and then that $\varphi_R \colon X \to Z$ is a $\mathbb{P}^{n-2}$-bundle, locally trivial in the complex topology.
Let $l(R) = n-2$ and assume φ is of fiber type; then, by 1.3.17 and 1.3.19, φ is onto either a smooth curve or a smooth surface or a threefold. If F is a general fiber the pair $(F, \mathcal{E}_{|F})$ is a del Pezzo manifold (F, L) with $\mathcal{E}_{|F} = L^{\oplus r}$ by 3) of 2.4.1 in the first case, and $(\mathbb{Q}^{n-2}, \mathcal{O}_{\mathbb{Q}^{n-2}}(1)^{\oplus r})$ by 2) of 2.3.1 in the second case. If $\dim Y = 3$ then it

is well known that Y has rational and Gorenstein singularities. Moreover in our case they are also isolated: to prove this take a general hyperplane section S in Y and consider the map $\varphi_{|\varphi^{-1}(S)}\colon \varphi^{-1}(S) \to S$. By [AnBaWi92, Proposition 1.3] this map is elementary and therefore, by 1.3.19, S is smooth, thus Y has isolated singularities. For the general fiber F the pair $(F, \mathcal{E}_{|F})$ is $(\mathbb{P}^{n-3}, \mathcal{O}_{\mathbb{P}^{n-3}}(1)^{\oplus r})$ by 2.2.1. As in the proof of [AnMe97, Theorem 5.1], the same holds for all fibers F over smooth points. We are left with the case $l(R) = n-1, n-2$, $\rho(X) > 1$ and $\varphi_R\colon X \to Y$ birational. In the first case, by [AnOc02b, Theorem 1.1], Y is smooth and φ_R is the blow-up of Y at a point. Moreover, if E denotes the exceptional locus of φ_R, by adjunction $\det \mathcal{E}_{|E} = \mathcal{O}_{\mathbb{P}^{n-1}}(r+a)$; in particular, if $r \leq n-2$, $\mathcal{E}_{|E} = \mathcal{O}_{\mathbb{P}^{n-1}}(2) \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(1)^{\oplus(n-3)}$. In the second case, by [AnOc02b, Theorem 5.3], φ_R is divisorial and, if E denotes the exceptional locus of φ_R: either $\varphi_R(E)$ is a point and $(E, -E_E) = (\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(2))$, then by adjunction $\det \mathcal{E}_{|E} = \mathcal{O}_{\mathbb{P}^{n-1}}(r)$, therefore $\mathcal{E}_{|E} = \mathcal{O}_{\mathbb{P}^{n-1}}(1)^{\oplus r}$ by 2.1.10; or $\varphi_R(E)$ is a point and $(E, -E_E) = (\mathbb{Q}^{n-1}, \mathcal{O}_{\mathbb{Q}^{n-1}}(1))$, where $\mathbb{Q}^{n-1}$ is a possibly singular hyperquadric, then by adjunction $\det \mathcal{E}_{|E} = \mathcal{O}_{\mathbb{Q}^{n-1}}(r)$, therefore $\mathcal{E}_{|E} = \mathcal{O}_{\mathbb{Q}^{n-1}}(1)^{\oplus r}$ by 2.1.10; or Y is smooth and φ_R is the blow-up along a smooth curve $\varphi_R(E) \subset Y$, and for all fibers $F \subset E$ by adjunction $\det \mathcal{E}_{|F} = \mathcal{O}_{\mathbb{P}^{n-2}}(r)$, therefore $(F, \mathcal{E}_{|F}) = (\mathbb{P}^{n-2}, \mathcal{O}_{\mathbb{P}^{n-2}}(1)^{\oplus r})$. *q.e.d.*

We note that the proposition is an improvement of [Ohpr, Propositions 0.8 and 0.9], where M. Ohno assumes $r \leq n-2$.

Remark 2.5.2. In the hypothesis of 2.5.1, assume that $K_X + \tau \det \mathcal{E}$ is big (and nef by the definition of τ). Then all rays in the face F defined by $K_X + \tau \det \mathcal{E}$ are not nef. If $\tau \geq \frac{n-2}{r}$ then they are described in 2.4.1 6) (see remark 2.4.3), 7), and 2.5.1 11),12). If moreover $\dim X \geq 4$ the exceptional loci of the rays in the face F are disjoint and therefore the map Φ associated to $K_X + \tau \det \mathcal{E}$ contracts them to different points and disjoint curves. The last statement follows from [Fu92, Theorem 2.4]. This allows to define the second reduction of the pair $(X, \mathcal{E})$ in the spirit of [BeSo95, Section 7].

Chapter 3

Comparing the cones of a variety and of an ample section

A classical approach of inductive type to the classification of smooth complex projective varieties X is that of slicing X with a number of general hyperplane sections, in order to find complex manifolds of smaller dimension, which are easier to describe. Then the variety X can be described by "lifting" the geometric properties of the new manifold. (*Apollonious method*).

This problem can be studied in the more general set up that was considered first by A. Lanteri and H. Maeda in [LaMa95], [LaMa96], [LaMa97]: let X be a smooth complex projective variety of dimension n and let $\mathcal{E}$ be an ample vector bundle of rank r on X such that there exists a section $s \in \Gamma(\mathcal{E})$ whose zero locus, $Z := (s)_0$, is a smooth submanifold of the expected dimension $\dim Z = \dim X - r = n - r$.

In the spirit of the minimal model program, the classification of smooth varieties is strictly connected to the study of the Kleiman-Mori cone $\overline{NE}(X)$ of X, i.e. the closure of the cone generated by effective 1-cycles of X.

If $\dim Z \geq 3$, by Sommese's version of Weak Lefschetz theorem (1.4.7), the natural inclusion $N_1(Z) \to N_1(X)$ and the natural restriction map $N^1(X) \to N^1(Z)$ are isomorphisms, while if $\dim Z = 2$ the map $N_1(Z) \to N_1(X)$ is surjective, and the map $N^1(X) \to N^1(Z)$ is injective.

It follows that, if $\dim Z \geq 3$, the cone $\overline{NE}(Z)$ of Z can be seen as a subset of the cone $\overline{NE}(X)$ of X.

In this chapter we study the relations between the Kleiman-Mori cone of the variety X and the Kleiman-Mori cone of its section Z, in order to compare the geometry of X and Z.

In their papers [AnOc99] and [AnOc02a], M. Andreatta and G. Occhetta investigated this problem. In particular, they gave a condition under which an extremal contrac-

tion of Z can be lifted to a contraction of X and a couple of examples showing that the two cones can be different.
We start this chapter, whose main part contains the results of [AnNoOcpr], by recalling those examples and giving a new one; we note that the assumptions $\mathcal{E}$ very ample and Z Fano manifold are not sufficient to give $\overline{NE}(Z) = \overline{NE}(X)$.
Therefore the problem is to find suitable condition under which (at least a part of) the polyhedral part of the Kleiman-Mori cones is the same. For this purpose we recall the results proved in [AnOc99] and in [Ocpr].
We give a necessary and sufficient condition for determining whether an extremal ray of X is an extremal ray of Z, too: the idea is to consider $R_X := \mathbb{R}^+[C]$, an extremal ray in $\overline{NE}(X)$ such that $-(K_X + \det\mathcal{E}) \cdot R_X > 0$ and to prove that the intersection of Z with non trivial fibers of the contraction associated to R_X is of positive dimension. We also give a slight improvement of this result in case $\mathcal{E}$ has rank 1, i.e. $\mathcal{E}$ is a line bundle.
Then we deal with the case in which Z does not have negative extremal rays, but the canonical bundle of Z is not ample. In this case K_Z turns out to be semiample and X admits a fibration in Fano varieties which extends the pluricanonical map of Z. If moreover $r = 1$ the effective cones of the general fibers of X and Z coincide, therefore this can be seen as a relative version of the results in section 3.2. Moreover we apply the previous results to the case in which Z is a surface with Kodaira dimension 0 or 1, not necessarily minimal, giving a different proof of the results obtained in [La96] and [LaMa01].
In the fourth section we restrict to the case in which $\mathcal{E}$ is an ample line bundle (and we change notation, denoting by L the ample line bundle $\mathcal{E}$).
We study the case in which $Z \in |L|$ is a Fano manifold of high index; namely $-K_Z = \mathrm{index}(Z)H_Z$, for some ample Cartier divisor H_Z on Z and $\mathrm{index}(Z) \geq \frac{\dim Z}{2}$. We prove that, if H_Z is spanned by global sections, then $\overline{NE}(Z) = \overline{NE}(X)$, apart from the case in which $Z \simeq \mathbb{P}^1 \times \mathbb{P}^3$, which gives rise to one of the example of the first section.
Then we apply our results to classify all the polarized varieties (X, L) in which Z is a Fano manifold of index $\dim Z - 2$ (resp. $\dim Z - 3$) of dimension ≥ 4 (resp ≥ 5), improving the results in [BeFaSo04].
Finally we study polarized varieties (X, L) in which $Z \in |L|$ is a Fano manifold whose anticanonical bundle $-K_Z = \det\mathcal{V}$, for some ample vector bundle $\mathcal{V}$ on Z of rank $r \geq \dim Z - 2$.

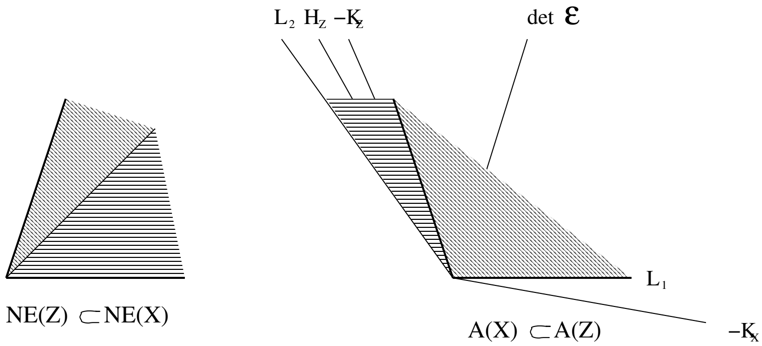
3.1 Examples

We start this chapter with the examples given in [AnNoOcpr, Section 2] which show that the inclusion $NE(Z) \subset NE(X)$ can be strict.

The first example comes from [AnOc99, Example 4.10], and generalizes an example of L. Bădescu (see also [LaMa95, Example 4.2]): consider the sequence

$$0 \longrightarrow \oplus^n \mathcal{O}_{\mathbb{P}^1} \longrightarrow \mathcal{G} := \oplus^n(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(s-a)) \longrightarrow \oplus^n \mathcal{O}_{\mathbb{P}^1}(s) \longrightarrow 0$$

which is exact in view of [Bă82, Remark 1, p.170] and choose a, s in such a way that $0 < a - s < a$.
The construction in [Ful84, B.5.6] applies and gives $\mathbb{P}^1 \times \mathbb{P}^{n-1}$ as a submanifold of $X = \mathbb{P}(\mathcal{G})$ which is the zero set of a general section of the ample vector bundle $\mathcal{E} = \oplus^n \xi_{\mathcal{G}}$.
Note that $\mathcal{E}$ is actually very ample and that for $n = 1$ it is a line bundle. The cones are described in the following picture



In particular there is a ray in common (the one associated to the curve contracted by the projections $p_1 : Z \to \mathbb{P}^1$).
In general the other rays are different, i.e. the contraction to the second factor of Z, $p_2 : Z \to \mathbb{P}^{n-1}$, cannot be extended to X.
Let L_i be the pull back through the projection p_i of the hyperplane bundle.
The contraction p_2 is supported by $K_Z + H_Z = bL_2 \quad (b > 0)$ and $H_Z = 2L_1 + (n+b)L_2$ is an ample line bundle on Z which is not the restriction of an ample line bundle on X.

Another example, suggested by Jaroslaw Wiśniewski, was given in [AnOc02a, Section 4], producing $Z = \mathbb{P}_{\mathbb{P}^2}(\mathcal{O}(1) \oplus \mathcal{O})$ as a section of an ample vector bundle on $X = \mathbb{P}^k \times \mathbb{P}^2$, $(k \geq 3)$.
In this case the ray corresponding to the projection on $\mathbb{P}^2$ is common to $NE(Z)$ and $NE(X)$, while the ray in $NE(Z)$ corresponding to the blow down on $\mathbb{P}^k$ lies in the interior of $NE(X)$.

A third example, suggested by Massimiliano Mella, is the following: let X be the blow up of the product $\mathbb{P}^r \times \mathbb{P}^1$ at a point x.
The cone of curves of X has three rays: $NE(X) =< s, f, e >$ where s is the class of the strict trasform of $\{x\} \times \mathbb{P}^1$, f is the class of the strict transform of a line through

x in the $\mathbb{P}^r$ which contains x and e is the class of a line in the exceptional divisor.
Let Z be a general section of a very ample line bundle. If Z is very ample of sufficiently high degree then Z does not contain any effective curve whose numerical class is s or f.
The first assertion is clear, since there is only a curve in the numerical class of s; to see the second consider the fiber of $p\colon X \to \mathbb{P}^1$ which contains the exceptional divisor. This fiber is reducible and consists of two components, namely the exceptional divisor, which is $\mathbb{P}^r$, and a $\mathbb{P}^1$-bundle over $\mathbb{P}^{r-1}$.
Any curve in X whose numerical class is f has to be a fiber of this last $\mathbb{P}^1$-bundle. But any section of a sufficiently high degree very ample line bundle does not contain such curves.
So $NE(Z)$ is a subcone of $NE(X)$ and there is only one ray in common, the one generated by e.

Remark 3.1.1. Note that these examples show that the assumptions $\mathcal{E}$ very ample and Z Fano variety are not sufficient to give $\overline{NE}(Z) = \overline{NE}(X)$.

3.2 Comparing the cones

3.2.1. Let X be a smooth complex projective variety of dimension n and let $\mathcal{E}$ be an ample vector bundle of rank r on X such that there exists a section $s \in \Gamma(\mathcal{E})$ whose zero locus $Z := (s)_0$ is a smooth submanifold of X of the expected dimension $\dim Z = \dim X - r = n - r$.

We begin with a result which gives a necessary and sufficient condition for determining whether an extremal ray of X is an extremal ray of Z, too.

Theorem 3.2.2. *[AnNoOcpr, Theorem 3.1] Let X, $\mathcal{E}$ and Z be as in 3.2.1. Assume that $R_X = \mathbb{R}^+[C]$ is an extremal ray of X such that $-(K_X + \det\mathcal{E}) \cdot C > 0$, then there is a curve on Z whose numerical class is $[\lambda C]$ for some $\lambda \in \mathbb{R}^+$.*
In particular, if $N_1(X) = N_1(Z)$ (for instance if $\dim Z \geq 3$), then an extremal ray of X is also an extremal ray of Z if and only if it stays in the semi-space defined by $\{x \in N_1(X) \mid -(K_X + \det\mathcal{E}) \cdot x > 0\}$.
In particular, if Z is not minimal, there exists at least one extremal ray which is in common between $\overline{NE}(X)$ and $\overline{NE}(Z)$.

Proof. Let $R_X = \mathbb{R}^+[C]$ be an extremal ray of X such that $-(K_X + \det\mathcal{E}) \cdot C > 0$, and let $\Phi\colon X \to Y$ be the associated contraction. In particular we have that the length of the ray R_X is $l(R_X) \geq \det\mathcal{E} \cdot C + 1 \geq r + 1$.

Let S be an irreducible component of a non trivial fiber of Φ. The theorem follows if we prove that $\dim Z \cap S \geq 1$, since in this case there is a curve whose numerical class is in R_X which stays in Z.
If Φ is birational, then $\dim S \geq l(R_X) \geq r+1$, by the inequality 1.3.17, and, by 1.4.6, $\dim Z \cap S \geq 1$. If Φ is of fiber type, then $\dim S \geq l(R_X) - 1 \geq r$, by the inequality 1.3.17. If $\dim S \geq r+1$, we conclude again by 1.4.6.
Therefore we can assume by contradiction that $\dim S = r$ and $\dim Z \cap S = 0$.
Let C be a rational curve in S; then $-K_X \cdot C > r$. On the other hand the length of the ray is $l(R) \leq \dim X - \dim Y + 1 = r+1$. Thus we get $-K_X \cdot C = r+1$ and $\det \mathcal{E} \cdot C = r$; in particular we have $-(K_X + \det \mathcal{E}) \cdot C = 1$.
Observe that $\dim F = \mathrm{rk}\mathcal{E}_F$ and $K_F + \det \mathcal{E}_F$ is not nef; then, by [YeZa90, Theorem 1], $(F, \mathcal{E}_F) \cong (\mathbb{P}^r, \oplus^r \mathcal{O}_{\mathbb{P}^r}(1))$.
By Fujita's characterization of scrolls, see [Fu85, Lemma 2.12], X is a $\mathbb{P}^r$-bundle over Y; in particular $\rho(X) = \rho(Y) + 1$. Since $\dim Z \cap F = 0$ and $\mathcal{E}_{|F} = \oplus^r \mathcal{O}_{\mathbb{P}^r}(1)$, Z is isomorphic to Y and this is a contradiction with the Lefschetz theorem.
The condition in the second part of the theorem is clearly necessary. *q.e.d.*

In the case of ample divisor we have a slightly better result:

Theorem 3.2.3. *[AnNoOcpr, Theorem 3.2] Let $X, \mathcal{E}$ and Z be as in 3.2.1. Assume that $\dim Z \geq 3$ and that $r = 1$, i.e. that Z is a section of an ample line bundle L. Then the extremal rays of X which stay in the closed semi-space defined by $\{x \in N_1(X) \mid -(K_X + L) \cdot x \geq 0\}$ are in the boundary of $\overline{NE}(Z)$ as well.*

Proof. We repeat the above proof for an extremal ray of X, $R_X = \mathbb{R}^+[C]$, such that $-(K_X + L) \cdot C \geq 0$ and with $\Phi \colon X \to Y$ its associated contraction. If $\dim F \cap Z > 0$, then the theorem follows as above, thus we can assume that $\dim F = 1$ and $\dim F \cap Z = 0$ for all irreducible components of non trivial fibers of Φ. Thus Φ is the blow up of a smooth subvariety or Φ is a conic bundle, see [An85]. Moreover, since $-(K_X + L) \cdot C \geq 0$, we have $L \cdot C \leq -K_X \cdot C$ which is 1 in the first case and 1 or 2 in the second. The result in this case follows from the [Ko89, Lemma]. *q.e.d.*

Corollary 3.2.4. *[AnNoOcpr, Corollary 3.3] Let $X, \mathcal{E}$ and Z be as in 3.2.1. Assume that $r = 1$. If $mK_Z = \mathcal{O}_Z$ for some $m > 0$, then X is Fano, $K_Z = \mathcal{O}_Z$ and $NE(X) = NE(Z)$.*

Remark 3.2.5. [AnNoOcpr, Remark 3.6] After the paper was written, we found out that the result in theorem 3.2.3 is also in [HaLiWa02, Theorem 4.3]. Note that in [HaLiWa02, Theorem 4.3] a slightly more general result is stated, though not proved:

the fact that every ray in $\overline{NE}(X)$ in $-(K_X + L) \geq 0$ is a ray of $\overline{NE}(Z)$ does not imply that $\overline{NE}_{-(K_X+L)\geq 0}(Z) = \overline{NE}_{-(K_X+L)\geq 0}(X)$.

Remark 3.2.6. [AnNoOcpr, Remark 3.7] The last theorem and corollary should be true also in the case $r > 1$. However it seems difficult to prove in the general case the technical results from [An85] and [Ko89] used in the proof of Theorem 3.2.3.

A first step in this direction is the following:

Proposition 3.2.7. *Let $X, \mathcal{E}$ and Z be as in* (3.2.1). *Assume that* $r \geq 2$ *and* $\dim Z \geq 3$. *If $mK_Z = \mathcal{O}_Z$, then*

1. *X is a Fano manifold with $-K_X = \det \mathcal{E}$ and Z has $K_Z = \mathcal{O}_Z$.*
2. *if X has only birational extremal rays, then $NE(X) = NE(Z)$.*

Proof. By assumption $\mathcal{O}_Z = mK_Z = m(K_X + \det \mathcal{E})_{|Z}$, so $\mathcal{O}_X = m(K_X + \det \mathcal{E})$, from which $-K_X$ is ample. Now, Pic$X$ has no torsion, $m = 1$ and $K_Z = (K_X + \det \mathcal{E})_{|Z} = \mathcal{O}_Z$. Then X is a Fano manifold with $-K_X = \det \mathcal{E}$ and $K_Z = \mathcal{O}_Z$.
Now, let $R = \mathbb{R}^+[C]$ be an extremal ray on X, with associated contraction $\Phi\colon X \longrightarrow Y$. If there is a fiber F such that $\dim F \cap Z \geq 1$, then R is an extremal ray of Z, too, and we are done.
Suppose that Φ is such that $\dim F \cap Z \leq 0$ for any non trivial fiber F. From $0 \geq \dim F \cap Z \geq \dim F - r$, we have $\dim F \leq r$ for any F. Moreover the length of the ray is $l(R) \geq r$.
Since Φ is birational, by 1.3.17 we find that $\dim F = r$ and $\dim F \cap Z = 0$ for any non trivial fiber. Let E be the exceptional locus of Φ. By 1.3.17, $n - 1 \geq \dim E \geq l(R) - \dim F + n - 1$, then Φ is divisorial; moreover $l(R) = r = \dim F$. Then, by [AnOc02b, Theorem 5.2], Φ is the blow-up of Y along a smooth subvariety V of codimension $l(R) + 1 = r + 1$. Now, $\pi := \Phi_{|E}\colon E \longrightarrow V$ is a $\mathbb{P}^r$-bundle. We have rk$\mathcal{E}_F = r = \dim F$ and $K_F + \det \mathcal{E}_F$ is not nef; then, by [YeZa90, Theorem 2], $(F, \mathcal{E}_F) \cong (\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1)^{\oplus r})$. Moreover, since $\dim Z_{|F} = 0$, Z_E is a section of π. Therefore $\mathcal{E}_{|E}$ is ample on E; by the Lefschetz theorem, $\rho(V) = \rho(Z_E) \geq \rho(E)$, against $\rho(E/V) = 1$. *q.e.d.*

Remark 3.2.8. We note that the first part of the previous proposition holds also when Z is a smooth surface, as shown in [La96, Lemma 1.1].

Remark 3.2.9. [AnNoOcpr, Remark 3.8] To decide whether an extremal ray of Z is an extremal ray of X seems to be more difficult. In the following we simply recall two partial results proved in [AnOc02a] and [Ocpr].

Theorem 3.2.10. *[AnNoOcpr, Theorem 3.9] Let $X, \mathcal{E}$ and Z be as in 3.2.1, and let $R_Z = \mathbb{R}^+[C]$ be an extremal ray of Z. Then R_Z is extremal also in X if one of the following holds:*

a) [AnOc02a, Theorem 3.2] There is an ample line bundle H on X and a positive real number τ such that $\overline{NE}(Z)$ is contained in the semi-space $K_X + \det \mathcal{E} + \tau H \geq 0$ and $(K_X + \det \mathcal{E} + \tau H) \cdot C = 0$.

b) [Ocpr, Proposition 5] There is a component $V_X \subset \mathrm{Hom}(\mathbb{P}^1, X)$ which contains C and such that:
1) $\mathrm{Locus}(V_X) = X$.
2) V_X is proper (i.e. an unsplit family).

3.3 Ample sections without extremal rays

In this section we deal with the case in which Z does not have extremal rays, but its canonical bundle is not ample.

Lemma 3.3.1. *Let $X, \mathcal{E}$ and Z be as in 3.2.1. Assume that Z is minimal, has Kodaira dimension $\kappa(Z) < \dim Z$ and dimension $\dim Z \geq 3$. Then K_X is not nef.*

Proof. Since $\kappa(Z) < \dim Z$, K_Z is not ample; then, by the Kleiman's criterion, there exists a cycle $\Gamma \in \overline{NE}(Z) \setminus \{0\}$ such that $K_Z \cdot \Gamma \leq 0$.
Observe that if $K_Z \cdot \Gamma < 0$, then there is an effective cycle Γ in the negative part of the cone of Z, contradicting the minimality of Z. Therefore there is a cycle $\Gamma \in \overline{NE}(Z) \setminus \{0\}$ such that $K_Z \cdot \Gamma = 0$.
Now, since $NE(Z) \subset NE(X)$, there exists $\Gamma \in \overline{NE}(X) \setminus \{0\}$ such that $(K_X + \det \mathcal{E}) \cdot \Gamma = 0$, from which we get $K_X \cdot \Gamma < 0$, with Γ effective curve on X. *q.e.d.*

In the next theorem we show that, if Z does not have negative extremal rays but its canonical bundle K_Z is not ample, then K_Z is semiample and X admits a fibration in Fano varieties which extends the pluricanonical map of Z. Moreover, if $r = 1$, the effective cones of the general fibers of X and Z coincide, so this can be viewed as a relative version of the results in section 3.2.

Theorem 3.3.2. *[AnNoOcpr, Theorem 5.1] Let $X, \mathcal{E}$ and Z be as in 3.2.1. Assume that Z is minimal and has Kodaira dimension $0 \leq \kappa(Z) < \dim Z$. Then:*

1. *$K_X + \det \mathcal{E}$ is nef (in particular it is semiample) but not big, i.e. $K_X + \det \mathcal{E}$ is a good supporting divisor of a Fano-Mori contraction $\Phi \colon X \longrightarrow Y$ of fiber type.*

2. *K_Z is semiample and Φ extends the pluricanonical map $\varphi_{|mK_Z|}$ for $m \gg 0$.*

3. *The general fiber F of the contraction Φ is a Fano manifold of pseudoindex $\geq r$ with $-K_F = \det \mathcal{E}_F$ and $K_F|_{Z\cap F} = \mathcal{O}_F|_{Z\cap F}$; in particular, if $mK_Z = \mathcal{O}_Z$, then X is a Fano manifold with $-K_X = \det \mathcal{E}$ and $K_Z = \mathcal{O}_Z$.*

Proof. Assume by contradiction that $K_X + \det \mathcal{E}$ is not nef. Then there exists an extremal ray $\mathbb{R}^+[C] \in \overline{NE}(X)$ such that $(K_X + \det \mathcal{E}) \cdot C < 0$; then, by 3.2.2, there is a curve $\Gamma \subset Z$ such that $[\Gamma] \in \mathbb{R}^+[C]$. In particular $K_Z \cdot \Gamma = (K_X + \det \mathcal{E}) \cdot \Gamma < 0$, against the minimality of Z.
On the other hand $K_X + \det \mathcal{E}$ is not ample, otherwise $K_Z = (K_X + \det \mathcal{E})_{|Z}$ would be ample; so $K_X + \det \mathcal{E}$ is a good supporting divisor for an extremal face in $\overline{NE}(X)$.
By the Kawamata-Shokurov base point free theorem, there exists a positive integer m such that $m(K_X + \det \mathcal{E})$ is spanned by global sections. Therefore also $m(K_X + \det \mathcal{E})_{|Z} = mK_Z$ is spanned by global sections.
Let $\Phi \colon X \longrightarrow Y$ be the map defined by $m(K_X + \det \mathcal{E})$ and $\varphi \colon Z \longrightarrow Z'$ the map defined by mK_Z. Taking $m \gg 0$ we can assume that they both have connected fibers, that Y and Z' are normal and that the following diagram is commutative.

$$\begin{tikzcd} X \arrow[d, "\Phi"'] & Z \arrow[l, "i"'] \arrow[dl, "\Phi_{|Z}"'] \arrow[d, "\varphi"] \\ Y & Z' \arrow[l, "\pi"] \end{tikzcd}$$

Since $\kappa(Z) < \dim Z$, the map φ is of fiber type; by the above diagram, $\Phi_{|Z}$ is of fiber type and we claim that Φ itself is of fiber type.
Assume by contradiction that Φ is birational and let E be the exceptional locus. By 1.3.17, $\dim F \geq r$ for all non trivial fibers; so, by 1.4.6, $\dim F \cap Z \geq 0$ and therefore $\Phi(E) \subseteq \Phi_{|Z}(Z)$.
On the other hand, since $\Phi_{|Z}$ is of fiber type, Z is contained in E and thus $\Phi_{|Z}(Z) \subseteq \Phi(E)$.
Then $\Phi(E) = \Phi_{|Z}(Z)$ and $\dim \Phi(E) = \dim \Phi_{|Z}(Z) < \dim Z = n - r$.
Since $\dim Y = n$, it is possible to find a subvariety $W' \subset Y$ such that $\dim W' = r$ and $W' \cap \Phi(E) = \emptyset$. But Φ is an isomorphism away from E, so $W := \Phi^{-1}(W')$ is a subvariety of X of dimension r such that its intersection with E is empty. Therefore $Z \cap W = \emptyset$, but this is a contradiction since, by 1.4.6, $\dim Z \cap W \geq \dim W - r = 0$.

Let F' be a general fiber of Φ; note that $\dim F' \cap Z \geq \dim X - \dim Y - r \geq 1$. So we can apply Lefschetz theorem to F' and $\mathcal{E}_{F'}$, obtaining $H^0(Z \cap F', \mathbb{Z}) \cong H^0(F', \mathbb{Z}) \cong \mathbb{Z}$. Using the Universal Coefficient Theorem, we get $H_0(Z \cap F') \cong \mathbb{Z}$, so $\Phi_{|Z}$ has connected fibers.
Therefore φ and $\Phi_{|Z}$ are morphisms with connected fibers onto normal varieties which contract curves in the same ray. This implies that π is an isomorphism.
Let F be the general fiber of Φ. We have that $K_F = K_X|_F = -\det \mathcal{E}|_F$; then F is a Fano manifold of pseudoindex $\geq r$. Moreover, if we consider the restriction of F to Z, we have $K_F|_{Z\cap F} = (K_X + \det \mathcal{E}|_F)_{Z \cap F} = \mathcal{O}_F|_{Z\cap F}$.
In particular, if $\mathcal{O}_Z = mK_Z = m(K_X + \det \mathcal{E})_Z$, then $\mathcal{O}_X = m(K_X + \det \mathcal{E})$ and thus $-K_X$ is ample. So X is a Fano manifold with $-K_X = \det \mathcal{E}$ and $K_Z = \mathcal{O}_Z$. *q.e.d.*

Remark 3.3.3. [AnNoOcpr, Remark 5.2] If Z is a minimal variety, then it is conjectured that $\kappa(Z) \geq 0$ (see [KaMaMa87] and [Mi88]). The conjecture is true for minimal surface or threefold.

Now, we consider the case in which Z is a surface of Kodaira dimension 0 or 1, without the assumption of minimality; this was done in [La96] and [LaMa01], here we give a different proof. In higher dimensions, even for $\dim Z = 3$, this is much more difficult.

Theorem 3.3.4. *[AnNoOcpr, Theorem 5.3] Let $X, \mathcal{E}$ and Z be as in 3.2.1. Assume that $r \geq 2$ and that Z is a surface of Kodaira dimension $\kappa(Z) = 0$ or an elliptic surface of Kodaira dimension $\kappa(Z) = 1$. Then $(X, \mathcal{E})$ is one of the following:*

1. *$X = \mathbb{P}_W(\mathcal{F})$, where $\mathcal{F}$ is an ample vector bundle of rank $n-1$ over a smooth surface W with $\kappa(W) = \kappa(Z)$ and $\mathcal{E} = \Phi^*\mathcal{V} \otimes \xi$, where ξ is the tautological line bundle on X, $\mathcal{V}$ is a vector bundle of rank $n-2$ on W and $\Phi \colon X \longrightarrow W$ is the bundle projection. In this case Z is not minimal and $\Phi|_Z \colon Z \longrightarrow W$ is a birational morphism, but not an isomorphism.*

2. *There exist a birational morphism $\Phi \colon X \longrightarrow X'$ expressing X as a projective manifold X' blown up at a finite set B of points (possibly empty) and an ample vector bundle $\mathcal{E}'$ of rank $n-2$ on X' such that $\mathcal{E} = \Phi^*\mathcal{E}' \otimes [-\Phi^{-1}(B)]$ and $K_{X'} + \det \mathcal{E}'$ is nef. In this case the triplet $(X', \mathcal{E}', Z' := \Phi(Z))$ is as in 3.2.1, $r \geq 2$, and Z' is a minimal surface with $\kappa(Z) = \kappa(Z')$.*

   Moreover:

   if $\kappa(Z) = 0$, then X' is a Fano manifold with $-K_{X'} = \det \mathcal{E}'$ and Z' is a $K3$

surface dominated by Z via the birational morphism $\Phi|_Z$;
if $\kappa(Z) = 1$, then X' is endowed with a morphism $\Phi\colon X' \longrightarrow Y$ onto a smooth curve Y, whose general fiber F is a projective manifold of dimension $n-1$ satisfying the condition $K_F + \det \mathcal{E}'_F = \mathcal{O}_F$; Φ induces on Z' the elliptic fibration.

Proof. If Z is minimal the proof follows easily from 3.3.2.
Assume therefore that Z is not minimal; each of the extremal rays of Z corresponds to the contraction of a (-1)-curve. Let H be an ample line bundle on X and let τ be the nefvalue of H_Z, i.e. the minimum real number such that $K_Z + \tau H_Z$ is nef. Then $K_X + \det \mathcal{E} + \tau H$ is nef but not ample and it is zero exactly on the curves of a face F which is extremal both in Z and in X.
Let $\Phi\colon X \longrightarrow W$ be the contraction associated to a ray in F and let $\varphi\colon Z \longrightarrow Z'$ the contraction of the (-1)-curve corresponding to the ray. By [AnOc02a, Proposition 3.8], Φ can be either birational or of fiber type; moreover in the last case it is an adjunction theoretic scroll onto W and $\Phi_{|Z} = \varphi$. Since $\dim W = 2$ we can apply [BeSo95, Proposition 14.1.3] which says that Φ is actually a $\mathbb{P}^{n-2}$-bundle and we are in the case 1. of the Theorem. Suppose now that Φ is birational. By [AnOc02a, Proof of Theorem 1.2] W is smooth, Φ is the blow-up of a point B on W and $\Phi_{|Z} = \varphi$. Moreover $\mathcal{E}_F \cong \oplus^r \mathcal{O}_{\mathbb{P}^{r-1}}(1)$, where F is a fiber of Φ. Then, by [AnOc02a, Lemma 2.9] there exists an ample vector bundle $\mathcal{E}'$ of rank $r = n-2$ on X' such that $\mathcal{E} \otimes \Phi^{-1}(B) = \Phi^*\mathcal{E}'$ and Z' is a section of $\mathcal{E}'$. If Z' is not minimal we can thus repeat the above arguments. We observe that the case of a fiber type contraction on X' cannot happen now and in any further steps and therefore we are in the case 2. of the Theorem. The claim can be proved exactly as in the last part of the proof of [AnOc02a, Theorem 1.4] *q.e.d.*

Remark 3.3.5. [AnNoOcpr, Remark 5.4] Lanteri and Maeda in [LaMa01] showed that the elliptic fibration $\Phi|_{Z'}\colon Z' \longrightarrow Y$ has actually no multiple fibers and the genus of the curve Y is $g(Y) = h^{1,0}(Z)$.

3.4 Adjunction

In this section we restrict to the case $r = 1$, i.e. $\mathcal{E}$ is an ample line bundle. In this case we change notation and we denote by L the ample line bundle $\mathcal{E}$. Therefore we will work in the following set-up:

3.4.1. Let L be an ample line bundle on a complex projective manifold X of dimension n, and let $Z \in |L|$ be a smooth submanifold of X.

The polarized varieties (X, L) such that $Z \in |L|$ is a Fano manifold of index $r \geq \dim Z - 2$ were studied by adjunction theory; in particular, the case of projective space and hyperquadrics was considered in [Bă81], the del Pezzo varieties were studied in [LaPaSo97] and the Mukai varieties were the object of the recent paper [BeFaSo04]. The next result we propose is an improvement of [BeFaSo04, Theorem 2.1], obtained combining the proof of that result with theorem 3.2.3.

Theorem 3.4.2. *[AnNoOcpr, Theorem 4.1] Let X, L and Z be as in 3.4.1 with* $\dim Z \geq 3$*. Assume that $-K_Z = qH_Z + B_Z$, where q is a positive integer, H_Z is ample and spanned on Z and B is nef on X (for instance $B = \mathcal{O}_X$). Assume also that $K_X + (q+1)L$ is nef and that $k(K_X + (q+1)L) \geq 2$. Then H is ample and X is Fano. Moreover $NE(X) = NE(Z)$.*

Proof. By assumption $-qH = K_X + L + B$ and therefore $q(L - H) = K_X + (q+1)L + B$. Our assumption, together with Kawamata base point freeness, say that $L - H$ is semiample and that $k(L - H) \geq 2$. Therefore by [ShSo85, Theorem 7.65] we have that $H^1(X, H - L) = 0$.
This vanishing, together with the long exact sequence associated to the sequence

$$0 \to H - L \to H \to H_Z \to 0,$$

gives that H is spanned on Z.
Since Z is ample, this implies that H is spanned out of a finite set of points, therefore H is nef.
On the other hand $-K_X = L + qH + B$ is therefore ample, i.e. X is a Fano manifold. Assume by contradiction that H is not ample, i.e. there exists an extremal ray $R_X = \mathbb{R}^+[C]$ on which H is zero. We consider the contraction associated to R_X. Note that, since $H \cdot C = 0$, we have that $-(K_X + L) \cdot C = B \cdot C \geq 0$ and R_X is an extremal ray of Z, by 3.2.3. Then there exists a curve $\Gamma \subset Z$ whose numerical class is in R_X such that $\Gamma \cdot H = 0$. Therefore H_Z is not ample, against the assumption.
Now, $-(K_X + L) \cdot x = (qH + B) \cdot x > 0$ for all $x \in N_1(X)$; therefore we conclude by 3.2.2 *q.e.d.*

The next result shows that in some cases we can avoid the assumptions on $K_X + (q+1)L$.

Theorem 3.4.3. *[AnNoOcpr, Theorem 4.2] Let X, L and Z be as in 3.4.1. Assume that Z is a Fano variety of dimension* $\dim Z \geq 4$ *and index r, i.e. $-K_Z = rH_Z$ for some (ample) Cartier divisor H_Z of Z. Assume that H_Z is spanned .*
If $r \geq \frac{\dim Z}{2}$, then X is a Fano variety and $NE(X) = NE(Z)$ unless $Z = \mathbb{P}^1 \times \mathbb{P}^3$ and X is a projective bundle over $\mathbb{P}^1$ or $Z = \mathbb{P}^1 \times V$, with (V, A) a del Pezzo threefold of Picard number one and pseudoindex 2, and X is a del Pezzo fibration over $\mathbb{P}^1$.

Proof. If $D := K_X + (r+1)L$ is nef and $\kappa(D) \geq 2$ the result follows from theorem 3.4.2.
So let us assume first that D is not nef, i.e. that $D \cdot C < 0$ for some effective curve C; in this case there exists an extremal ray $R = \mathbb{R}^+[\Gamma]$ such that $D \cdot R < 0$.
In particular $-(K_X + L) \cdot R = (-D + rL) \cdot \Gamma > 0$, so that, by theorem 3.2.3, R is extremal for $NE(Z)$.
Let $\Gamma_Z \subset Z$ be a minimal extremal curve in R; i.e. a rational curve such that $[\Gamma_Z] \in R$ for which $-K_Z \cdot \Gamma_Z$ is minimal. By Mori theory it is known that $-K_Z \cdot \Gamma_Z \leq \dim Z + 1$; moreover equality holds if and only if $Z \simeq \mathbb{P}^{\dim Z}$, by [ChMiSB01]. We can assume that the last is not the case (otherwise $X \simeq \mathbb{P}^n$ and the theorem is obvious) so $-K_Z \cdot \Gamma_Z \leq \dim Z$.
Since $D = r(L - H)$ we have that $(L - H) \cdot \Gamma_Z < 0$, hence $H \cdot \Gamma_Z \geq 2$.
Therefore

$$2r \leq rH_Z \cdot \Gamma_Z = -K_Z \cdot \Gamma_Z \leq \dim Z,$$

forcing $r \leq \frac{n-1}{2}$.
We are thus left with the case $r = \frac{n-1}{2}$; in this case Z has an extremal ray of length $= \dim Z$; note also that $L \cdot \Gamma_Z = 1$, so $K_Z + (n-1)L_Z$ is not ample.
By the first step of adjunction theory, see [BeSo95, Section 7.2], either Z is a hyperquadric or Z is a projective bundle over a curve, but the first is impossible since $r = \frac{n-1}{2} \neq \dim Z$.
In the second case Z, being a Fano variety and a projective bundle over a curve can be only a product $\mathbb{P}^1 \times \mathbb{P}^{n-2}$ or the blow up of $\mathbb{P}^{n-1}$ along a codimension two linear subspace; in the first case we have $r = 2$, so $\dim Z = 4$, while in the second we have $r = 1$ and $\dim Z = 2$ against our assumptions.
The description of X now follows from [AnOc99, Proposition 4.9 and Remark 4.11].

We can thus assume that D is nef and that $\kappa(D) \leq 1$.
If $\kappa(D) = 0$ we have

$$-K_X \equiv (r+1)L.$$

In this case X is a Fano variety and $-(K_X + L) = rL$ is ample. We can apply theorem 3.2.3 to get $NE(X) = NE(Z)$ and we are done.
If $\kappa(D) = 1$ then any extremal ray of Z not contracted by D has fibers of dimension ≤ 1. This implies that $r = 2$ and thus that $\dim Z = 4$. By [Wi90b, Lemma 1.4] Z is $\mathbb{P}^1 \times \mathbb{P}^3$ or $\mathbb{P}^1 \times V$, with (V, A) a del Pezzo threefold (with respect to some ample line bundle A) with Picard number one and pseudoindex 2.
Let $\varphi \colon X \to C$ be the contraction associated to $K_X + 3L$; first of all note that, since Z is a Fano variety, then $C \simeq \mathbb{P}^1$.
For a general fiber F we have $-K_F = 3L_F$, so that (F, L_F) is a del Pezzo variety and $L_F = \mathcal{O}_F(1)$, so $Z \simeq \mathbb{P}^1 \times \mathbb{P}^3$ cannot be an ample section of such an X.

q.e.d.

We are now in the position to give a list of polarized varieties (X, L) as above with $Z \in |L|$ a Mukai manifold of dimension ≥ 4, strengthening the results in [BeFaSo04]; note that, apart from case 6 (d), all cases are effective.

Theorem 3.4.4. *[AnNoOcpr, Theorem 4.3] Let X, L and Z be as in 3.4.1. Assume that $\dim Z \geq 4$ and that (Z, H_Z) is a Mukai manifold, i.e. $-K_Z = (n-3)H_Z$, for some ample line bundle H_Z on Z.*
Then the triple (X, L, H) is one of the following:

1. *(X, L) is a Mukai manifold, i.e. $-K_X = (n-2)L$, with $\rho = 1$ and $L = H$.*

2. *(X, H) is a del Pezzo manifold, i.e. $-K_X = (n-1)H$, with*
   $\rho = 1$ and $L = 2H$.

3. *$X = \mathbb{Q}^n$, a hyperquadric in $\mathbb{P}^{n+1}$, $L = \mathcal{O}_{\mathbb{Q}^n}(3)$ and $H = \mathcal{O}_{\mathbb{Q}^n}(1)$.*

4. *$X = \mathbb{P}^n$, $L = \mathcal{O}_{\mathbb{P}^n}(4)$ and $H = \mathcal{O}_{\mathbb{P}^n}(1)$.*

5. *$\dim X = 6$ and*

   (a) *$X = \mathbb{P}^6$, $L = \mathcal{O}_{\mathbb{P}^6}(1)$ and $H = \mathcal{O}_{\mathbb{P}^6}(2)$.*
   (b) *$X = \mathbb{P}^3 \times \mathbb{P}^3$ and $L = H = \mathcal{O}(1,1)$, i.e. $Z = \mathbb{P}(T_{\mathbb{P}^3})$.*

6. *$\dim X = 5$ and*

   (a) *$X = \mathbb{Q}^5$, a hyperquadric in $\mathbb{P}^6$, $L = \mathcal{O}_{\mathbb{Q}^5}(1)$ and $H = \mathcal{O}_{\mathbb{Q}^5}(2)$.*
   (b) *$X = \mathbb{P}^5$, $L = \mathcal{O}_{\mathbb{P}^5}(2)$ and $H = \mathcal{O}_{\mathbb{P}^5}(2)$.*
   (c) *X is a projective bundle over $\mathbb{P}^1$ and $Z = \mathbb{P}^1 \times \mathbb{P}^3$ (see section two; X is not necessarily Fano and $NE(X) \neq NE(Z)$).*
   (d) *X is a del Pezzo fibration over $\mathbb{P}^1$ and $Z = \mathbb{P}^1 \times V$, with (V, A) a del Pezzo threefold of Picard number one and pseudoindex 2.*
   (e) *$X = \mathbb{P}^2 \times \mathbb{P}^3$, $L = \mathcal{O}(1,2)$ and $H = \mathcal{O}(1,1)$.*
   (f) *(X, L) is a Mukai 5-fold, i.e. $-K_X = 3L$, (with $\rho = 2$) and $H = L$. According to [Wi91b] they are:*
      i. *$X = \mathbb{P}^2 \times \mathbb{Q}^3$, $L = H = \mathcal{O}(1,1)$.*
      ii. *$X = \mathbb{P}(T_{\mathbb{P}^3})$, $L = H = \mathcal{O}(1,1)$.*

iii. $X = \mathbb{P}_{\mathbb{P}^3}(\mathcal{O}_{\mathbb{P}^3}(2) \oplus \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 2})$, $L = H = \xi + p^*\mathcal{O}_{\mathbb{P}^3}(1)$.

Proof. The case $\rho := \rho(X) = \rho(Z) = 1$ is straightforward since in this case H is ample on X and we have the equality $-K_X = (n-3)H + L$ (for more details one can look at the proof of [BeFaSo04, Proposition 3.1]). We can thus assume $\rho := \rho(X) = \rho(Z) > 1$.
By the theorems in [Wi90a] applied to Z, we have that

$$n - 3 \leq \frac{\dim Z + 2}{2} = \frac{n+1}{2}$$

with equality if and only if $Z = \mathbb{P}^3 \times \mathbb{P}^3$.
Since the last cannot be an ample section of any projective manifold, (see [BeSo95, Corollary 5.2.4]), we have that $n \leq 6$.
If $n = 6$ then, by [Wi91b] we have the following possibilities for Z:

1. $Z = \mathbb{P}^2 \times \mathbb{Q}^3$.
2. $Z = \mathbb{P}(T_{\mathbb{P}^3})$, $H_Z = \mathcal{O}(1,1)$.
3. $Z = \mathbb{P}_{\mathbb{P}^3}(\mathcal{O}_{\mathbb{P}^3}(2) \oplus \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 2})$.

Only $Z = \mathbb{P}(T_{\mathbb{P}^3})$ is an ample divisor in a smooth X. In fact, the first case is ruled out by [BeSo95, Corollary 5.2.4] and the third case is ruled out by [Fu80, Theorem 5.8].
Note that H_Z is spanned so we can apply theorem 3.4.2 which gives that H is ample. By [AnOc99, Theorem 4.1] each contraction on $Z = \mathbb{P}(T_{\mathbb{P}^3})$ lifts to a $\mathbb{P}^3$-bundle on X. It is straightforward to prove now that $X = \mathbb{P}^3 \times \mathbb{P}^3$ and that $L = H = \mathcal{O}(1,1)$.
Let then $n = 5$. By [Wi90b, 2.6] H_Z is spanned unless $Z = \mathbb{P}^1 \times V_1$, where (V_1, A) is a del Pezzo threefold (with respect to some ample line bundle A) of degree one in this case we repeat the final part of the proof of theorem 3.4.3 and we get that X is a del Pezzo fibration over $\mathbb{P}^1$ (it cannot be a projective bundle or a quadric fibration because $-K_{V_1}$ is not spanned).
So we can assume that H_Z is spanned and apply theorem 3.4.3; in particular, if X is not as in 6.(c) or in 6.(d) then X is a Fano variety and $NE(X) = NE(Z)$.
It follows that the line bundle H which restricts to H_Z is ample. Note that, by inequality 1.3.17, Z and X cannot have small contractions. Note also that $-K_X = 2H + L$. Assume that $H \cdot C_1 \neq L \cdot C_1$. In particular $l(R_1) \geq 4$. We claim that there are two fibers F_1 and F_2 of the contractions of R_1 and R_2 which have nonempty intersection. If both the contractions are of fiber type this is clear. Otherwise, if R_1 is a birational contraction, its exceptional locus E_1 is an effective divisor, and has positive intersection with at least one extremal ray (this is a general fact on Fano manifolds). Since $E_1 \cdot R_1 < 0$ we have $E_1 \cdot R_2 > 0$ and the claim is proved.

Let F_1 and F_2 the two fibers with a point in common; we have $\dim F_1 + \dim F_2 \leq 5$, $\dim F_1 \geq l(R_1) - 1 \geq 3$ and $\dim F_2 \geq l(R_2) - 1 = 2$.
Therefore both the contractions are of fiber type and the preceding inequalities are true for any fiber.
In particular, by [Fu85, Lemma 2.12], φ_1 is a $\mathbb{P}^3$-bundle and φ_2 is a $\mathbb{P}^2$-bundle.
Now, by [Sa85, Theorem A], $X = \mathbb{P}^2 \times \mathbb{P}^3$ and that $L = \mathcal{O}(1,2)$ and $H = \mathcal{O}(1,1)$.
Therefore we can assume now that $L = H$, i.e. (X, L) is a Mukai 5-fold with $\rho = 2$ and the result follows again from [Wi91b]. *q.e.d.*

Now we want to investigate polarized varieties (X, L) as above with $Z \in |L|$ a Fano manifold of index $\dim Z - 3$ and dimension ≥ 5.

Theorem 3.4.5. *Let X, L and Z be as in 3.4.1. Assume that $\dim Z \geq 5$ and that Z is a Fano manifold of index $\dim Z - 3$. Assume that $-K_Z = (n-4)H_Z$ for some ample and spanned line bundle H_Z on Z.*
Then one of the following holds:

1. *X is a Fano manifold of index $n-3$, i.e. $-K_X = (n-3)H$, with $\rho = 1$ and $L = H$.*

2. *(X, H) is a Mukai manifold, i.e. $-K_X = (n-2)H$, with $\rho = 1$ and $L = 2H$.*

3. *(X, H) is a del Pezzo manifold, i.e. $-K_X = (n-1)H$, with $\rho = 1$ and $L = 3H$.*

4. *$X = \mathbb{Q}^n$, a hyperquadric in $\mathbb{P}^{n+1}$, $L = \mathcal{O}_{\mathbb{Q}^n}(4)$ and $H = \mathcal{O}_{\mathbb{Q}^n}(1)$.*

5. *$X = \mathbb{P}^n$, $L = \mathcal{O}_{\mathbb{P}^n}(5)$ and $H = \mathcal{O}_{\mathbb{P}^n}(1)$.*

6. $\dim X = 8$ *and*

   (a) *$X = \mathbb{P}^8$, $L = \mathcal{O}_{\mathbb{P}^8}(1)$ and $H = \mathcal{O}_{\mathbb{P}^8}(2)$.*

   (b) *$X = \mathbb{P}^4 \times \mathbb{P}^4$ and $L = H = \mathcal{O}(1,1)$, i.e. $Z = \mathbb{P}(T_{\mathbb{P}^4})$.*

7. $\dim X = 7$ *and*

   (a) *$X = \mathbb{Q}^7$, a hyperquadric in $\mathbb{P}^8$, $L = \mathcal{O}_{\mathbb{Q}^7}(1)$ and $H = \mathcal{O}_{\mathbb{Q}^7}(2)$.*

   (b) *$X = \mathbb{P}^7$, $L = \mathcal{O}_{\mathbb{P}^7}(2)$ and $H = \mathcal{O}_{\mathbb{P}^7}(2)$.*

   (c) *$X = \mathbb{P}^3 \times \mathbb{Q}^4$, $L = H = \mathcal{O}(1,1)$.*

(d) $X = \mathbb{P}_{\mathbb{P}^4}(T_{\mathbb{P}^4})$, *i.e. a smooth divisor of bidegree* $(1,1)$ *in* $\mathbb{P}^4 \times \mathbb{P}^4$, $L = H = \mathcal{O}(1,1)$.

(e) $X = \mathbb{P}_{\mathbb{P}^4}(\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus 3})$, *i.e. the blow-up of* $\mathbb{P}^7$ *along a* $\mathbb{P}^2$, $L = H = \mathcal{O}(1,1)$.

(f) $X = \mathbb{P}^4 \times \mathbb{P}^3$, $L = \mathcal{O}(2,1)$ *and* $H = \mathcal{O}(1,1)$.

8. $\dim X = 6$ *and*

(a) (X, L) *is a del Pezzo manifold with* $\rho = 1$ *and* $H = 2L$.

(b) $X = \mathbb{Q}^6$, *a hyperquadric in* $\mathbb{P}^7$, $L = \mathcal{O}_{\mathbb{Q}^6}(2)$ *and* $H = \mathcal{O}_{\mathbb{Q}^6}(2)$.

(c) $X = \mathbb{P}^6$, $L = \mathcal{O}_{\mathbb{P}^6}(3)$ *and* $H = \mathcal{O}_{\mathbb{P}^6}(2)$.

(d) $X = \mathbb{P}^6$, $L = \mathcal{O}_{\mathbb{P}^6}(1)$ *and* $H = \mathcal{O}_{\mathbb{P}^6}(3)$.

(e) $X = \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ *and* $L = H = \mathcal{O}(1,1,1)$.

(f) $X = \mathbb{P}^3 \times \mathbb{P}^3$, $L = \mathcal{O}(2,2)$ *and* $H = \mathcal{O}(1,1)$.

(g) $X = \mathbb{P}^2 \times \mathbb{P}^4$, $L = \mathcal{O}(1,3)$ *and* $H = \mathcal{O}(1,1)$.

(h) $X = \mathbb{P}^3 \times \mathbb{Q}^3$, $L = \mathcal{O}(2,1)$ *and* $H = \mathcal{O}(1,1)$.

(i) $X = \mathbb{P}^2 \times \mathbb{Q}^4$, $L = \mathcal{O}(1,2)$ *and* $H = \mathcal{O}(1,1)$.

(j) X *is a Mukai fibration over* $\mathbb{P}^1$.

(k) X *is a hyperquadric fibration over* $\mathbb{P}^1$.

(l) X *is a scroll over a smooth surface.*

(m) $X = Bl_{\mathbb{P}^2}\mathbb{P}^6$.

(n) $X = Bl_{\mathbb{P}^1}\mathbb{P}^6$.

(o) X *is a Fano manifold of index* 3 *and Picard number* 2. *A classification is given in [Wi93, Theorems I and II].*

(p) X *has a structure of* $\mathbb{P}^3$*-bundle onto a smooth* 3*-fold* W *and has a structure of scroll onto a normal* 4*-fold* Y.

(q) X *has a structure of* $\mathbb{P}^2$*-bundle onto a smooth* 4*-fold* W *and has a structure of scroll onto a normal* 3*-fold* Y.

Proof.
Assume that $\rho := \rho(X) = \rho(Z) = 1$. Since, by the Lefschetz theorem, $\mathrm{Pic}(X) \cong \mathrm{Pic}(Z)$, H_Z extends to an ample line bundle H on X.

Assume that H_Z generates $\mathrm{Pic}(Z)$; then H generates $\mathrm{Pic}(X)$.
Then there exists $t \in \mathbb{Z}_{\geq 1}$ such that $L = tH$ and $L_Z = tH_Z$.
So we have $-(n-4)H_Z = K_Z = (K_X + L)_{|Z}$, from which we deduce $-K_X = (n-4+t)H$.
Now, since the index of X cannot be greater than $n+1$, we find that $t \leq 5$.
If $t = 1$, then $-K_X = (n-3)H$, i.e. X is a Fano manifold of index $n-3$, and $L = H$.
If $t = 2$, then $-K_X = (n-2)H$, i.e. (X, H) is a Mukai manifold and $L = 2H$.
If $t = 3$, then $-K_X = (n-1)H$, i.e. (X, H) is a del Pezzo manifold and $L = 3H$.
If $t = 4$, then $-K_X = nH$, i.e. $(X, L) \cong (\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(4))$ and $H = \mathcal{O}_{\mathbb{Q}^n}(1)$.
If $t = 5$, then $-K_X = (n+1)H$, i.e. $(X, L) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(5))$ and $H = \mathcal{O}_{\mathbb{P}^n}(1)$.

Now we can assume that $\mathrm{Pic}(Z) \cong \mathbb{Z}[D_Z]$ for some ample line bundle D_Z on Z different from H_Z. Then $\mathrm{Pic}(X) \cong \mathbb{Z}[D]$ for some ample divisor D on X such that $D_{|Z} = D_Z$.
Then there exists $t \in \mathbb{Z}_{\geq 1}$ such that $L = tD$ and there exists $k \in \mathbb{Z}_{\geq 2}$ such that $H_Z = kD_Z$.
Since $-K_Z = k(n-4)D_Z$, by

$$2(n-4) \leq k(n-4) \leq \dim Z + 1 = n$$

we have that $n \leq 8$.
If $n = 8$, we have that $k = 2$; then $-K_Z = 8D_Z$. So we find $-K_X = (8+t)D$ and $t = 1$. Then $(X, L) \cong (\mathbb{P}^8, \mathcal{O}_{\mathbb{P}^8}(1))$ and $H = \mathcal{O}_{\mathbb{P}^8}(2)$.
If $n = 7$, we have that $k = 2$; then $-K_Z = 6D_Z$. So we find $-K_X = (6+t)D$ and either $t = 1$, in which case $(X, L) \cong (\mathbb{Q}^7, \mathcal{O}_{\mathbb{Q}^7}(1))$ and $H = \mathcal{O}_{\mathbb{Q}^7}(2)$, or $t = 2$, in which case $(X, L) \cong (\mathbb{P}^7, \mathcal{O}_{\mathbb{P}^7}(2))$ and $H = \mathcal{O}_{\mathbb{P}^7}(2)$.
If $n = 6$, we have that either $k = 2$ or $k = 3$.
If $k = 2$, then $-K_Z = 4D_Z$ and $-K_X = (4+t)D$. Then either $t = 1$, in which case (X, L) is a del Pezzo manifold and $H = 2L$, or $t = 2$, in which case $(X, L) \cong (\mathbb{Q}^6, \mathcal{O}_{\mathbb{Q}^6}(2))$ and $H = \mathcal{O}_{\mathbb{Q}^6}(2)$, or $t = 3$, in which case $(X, L) \cong (\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(3))$ and $H = \mathcal{O}_{\mathbb{P}^6}(2)$. If $k = 3$, then $-K_Z = 6D_Z$. So we find $-K_X = (6+t)D$ and $t = 1$. Then $(X, L) \cong (\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(1))$ and $H = \mathcal{O}_{\mathbb{P}^6}(3)$.

We can thus assume $\rho := \rho(X) = \rho(Z) > 1$.
By the theorems in [Wi90a] applied to Z, we have that

$$n - 4 \leq \frac{\dim Z + 2}{2} = \frac{n+1}{2}, \quad \text{i.e. } n \leq 9.$$

If the equality holds, $Z = \mathbb{P}^4 \times \mathbb{P}^4$, that cannot be an ample section of any projective manifold, (see [BeSo95, Corollary 5.2.4]). Then we have that $n \leq 8$.

If $n = 8$ then, by [Wi91b], we have the following possibilities for Z:

1. $Z = \mathbb{P}^3 \times \mathbb{Q}^4$.
2. $Z = \mathbb{P}(T_{\mathbb{P}^4})$, $H_Z = \mathcal{O}(1,1)$.
3. $Z = \mathbb{P}_{\mathbb{P}^4}(\mathcal{O}_{\mathbb{P}^4}(2) \oplus \mathcal{O}_{\mathbb{P}^4}(1)^{\oplus 3})$.

Only $Z = \mathbb{P}(T_{\mathbb{P}^4})$ is an ample divisor in a smooth X. In fact, the first case is ruled out by [BeSo95, Corollary 5.2.4] and the third case is ruled out by [Fu80, Theorem 5.5].
Note that H_Z is spanned so we can apply theorem 3.4.2 which gives that H is ample. By [AnOc99, Theorem 4.1] each contraction on $Z = \mathbb{P}(T_{\mathbb{P}^4})$ lifts to a $\mathbb{P}^4$-bundle on X. Therefore, by [Sa85, Theorem A], $X = \mathbb{P}^4 \times \mathbb{P}^4$ and $L = H = \mathcal{O}(1,1)$.

Assume now that $n \leq 7$. By assumption, n can be either 7 or 6. By [Wi90a, Theorem A], since the pseudoindex is an integral multiple of the index, we have that the pseudoindex of Z is equal to $n - 4$.
By [ChOcpr, Theorems 1.1 and 1.2], at least one of the extremal rays of Z is of fiber type: indeed Z has only birational contractions if it is a blow-up of $\mathbb{P}^5$ along a surface; by [Fu80, Theorem 5.5], this cannot be ample section of any manifold.

If $\dim X = 7$, then by [AnChOc04, Theorem 1.1], $\rho \leq 3$, with equality if and ony if $Z = \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$, which cannot be an ample section in any manifold, by [BeSo95, Corollary 5.2.4]. Therefore we are left with the case $\rho = 2$.
Since H_Z is spanned, by 3.4.3 X is Fano and $NE(X) = NE(Z)$. Therefore H, whose restriction to Z is H_Z, is ample. Moreover $-K_X = 3H + L$, thus the pseudoindex of X is ≥ 4.
Now, since Z has at least one extremal ray of fiber type and $NE(X) = NE(Z)$, X has an extremal ray of fiber type too, by [AnOc99]. Moreover, by [AnChOc04, Theorem 1.1] and the previous inequality, the pseudoindex of X is $4 = \dim X - 3$. By the list of [ChOcpr, Theorem 1.1], X can have either two contraction of fiber type or one contraction of fiber type and the other birational, contracting a divisor to a surface.
We observe that there exist two fibers F_1 and F_2 of the contractions Φ_{R_1} of R_1 and Φ_{R_2} of R_2 which have nonempty intersection.
Assume that $L = H$; then index$(X) = 4$. Therefore, by [Wi91b], we find the cases 7.(c), 7.(d) and 7.(e) of our proposition.
Assume now that $L \neq H$. Then there exists a curve C_1 such that $L \cdot C_1 \neq H \cdot C_1$. Put $R_1 := \mathbb{R}^+[C_1]$. Clearly $l(R_1) \geq 5$.
Now, R_1 is of fiber type. Indeed if this is not the case, by 1.3.17 we find that the

contraction Φ_{R_1} contracts a divisor to either a point or a curve, which is a contradiction with the type of rays of X. Moreover $\dim F_1 \geq 4$ for any fiber of the contraction Φ_{R_1}.
Now, we take F_1 and F_2 fibers of Φ_{R_1} and Φ_{R_2} respectively such that they have a point in common. Since $\dim F_1 + \dim F_2 \leq 7$ and $\dim F_1 \geq 4$, then $\dim F_2 \leq 3$. By 1.3.17, Φ_{R_2} is of fiber type. Moreover $l(R_2) = 4$ and the previous inequalities are actually equalities.
Therefore Φ_{R_1} is an equidimensional fibration over a normal 3-fold with general fiber $\mathbb{P}^4$ and Φ_{R_2} is an equidimensional fibration over a normal 4-fold with general fiber $\mathbb{P}^3$. Since $H \cdot C_i = 1$, where C_i is a minimal curve in R_i, by [BeSo95, Proposition 3.2.1] Φ_{R_1} gives to X a structure of $\mathbb{P}^4$-bundle and Φ_{R_2} gives to X a structure of $\mathbb{P}^3$-bundle. By [Sa85, Theorem A], $X = \mathbb{P}^4 \times \mathbb{P}^3$, $L = \mathcal{O}(2,1)$ and $H = \mathcal{O}(1,1)$.

Assume now that $\dim X = 6$ and put $D := K_X + 3L$.

Let us assume first that D is not nef, i.e. that $D \cdot C < 0$ for some effective curve C; in this case there exists an extremal ray $R = \mathbb{R}^+[\Gamma]$ such that $D \cdot R < 0$.
In particular $-(K_X + L) \cdot R = (-D + 2L) \cdot \Gamma > 0$, so that, by theorem 3.2.3, R is extremal for $NE(Z)$.
Let $\Gamma_Z \subset Z$ be a minimal extremal curve in R; i.e. a rational curve such that $[\Gamma_Z] \in R$ for which $-K_Z \cdot \Gamma_Z$ is minimal. By Mori theory it is known that $-K_Z \cdot \Gamma_Z \leq \dim Z + 1$; moreover, by [ChMiSB01], equality holds if and only if $Z \simeq \mathbb{P}^{\dim Z}$, contradicting $\rho > 1$. Therefore $-K_Z \cdot \Gamma_Z \leq \dim Z = 5$.
Since $D = 2(L - H)$ we have that $(L - H) \cdot \Gamma_Z < 0$, hence $H \cdot \Gamma_Z \geq 2$.
Therefore $4 \leq 2H_Z \cdot \Gamma_Z = -K_Z \cdot \Gamma_Z \leq \dim Z = 5$, which gives $K_Z \cdot \Gamma_Z = 4$, $H \cdot \Gamma_Z = 2$ and $L \cdot \Gamma_Z = 1$.
Then Z has an extremal ray of length $= \dim Z - 1 = 4$, that, by 1.3.17, can be either of fiber type or divisorial, contracting a divisor to a point.
Assume that the ray is divisorial. Then, by [BoCaWi], Z can be one of the following:

1. the blow-up of $\mathbb{P}^5$ in a point;
2. the blow-up of $\mathbb{Q}^5$ in a point;
3. the blow-up of Z' in a point P, where Z' is the blow-up of $\mathbb{P}^5$ along a smooth 3-fold of degree d, with $1 \leq d \leq 5$ and contained in a hyperplane H such that $P \notin H$.

In the first case, Z cannot be an ample section in any manifold, by [Fu80, Proposition 5.8]; in the other cases the index of Z is not 2.
Therefore R is of fiber type, note that $L \cdot \Gamma_Z = 1$, so $K_Z + (\dim Z - 1)L_Z$ is not ample.
Then either Z is a hyperquadric fibration over a smooth curve, or Z is a scroll over

a normal surface.
In the first case, by [AnOc99, Theorem 5.1], X is a hyperquadric fibration over a smooth curve. Moreover, since it is dominated by Z, which is a Fano manifold, the curve is $\mathbb{P}^1$. In the second case, by [AnOc99, Theorem 4.1] and 1.3.19, X is a scroll over a smooth surface.

We can thus assume that D is nef.

If $\kappa(D) = 0$ we have $-K_X \equiv 3L$. In this case X is a Fano manifold and $-(K_X + L) \equiv 2L$ is ample. We can apply theorem 3.2.3 to get $NE(X) = NE(Z)$.
Now the pseudoindex of X is at least 3; since the pseudoindex of Z is 2, there exists a rational curve Γ in Z such that $(K_X + L) \cdot \Gamma = 2$. Therefore, the pseudoindex and the index of X are 3. These varieties are classified in [Wi93, Theorems I and II]. In particular if $\rho = 3$ then $X = \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ and $L = H = \mathcal{O}(1,1,1)$.

If $\kappa(D) = 1$, let $\varphi\colon X \to C$ be the contraction associated to $K_X + 3L$; first of all note that, since Z is a Fano variety, then $C \simeq \mathbb{P}^1$.
For a general fiber F we have $-K_F = 3L_F$, so that (F, L_F) is a Mukai manifold. Therefore X is a Mukai fibration over $\mathbb{P}^1$.

If $\kappa(D) \geq 2$ then, by 3.4.2, X is a Fano manifold and $\overline{NE}(X) = \overline{NE}(Z)$.
Then, by [ChOcpr, Theorem 1.1], the Picard number is $2 \leq \rho \leq 3$.
Now, since $-K_X = 2H + L$, the pseudoindex of X is at least 3.
If $\rho = 3$, by [AnChOc04, Theorem 1.1], $X = \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$, and $L = H = \mathcal{O}(1,1,1)$, contradicting $\kappa(D) \geq 2$.
We can thus assume that $\rho = 2$. By [AnChOc04, Theorem 1.1], the pseudoindex of X is ≤ 4; moreover, if the equality holds, $X = \mathbb{P}^3 \times \mathbb{P}^3$, $L = \mathcal{O}(2,2)$ and $H = \mathcal{O}(1,1)$.
We can thus assume that the pseudoindex of X is $= 3$.
We note that $L \neq H$, otherwise $\kappa(D) = 0$.
Then there exists an extremal ray $R := \mathbb{R}^+[\Gamma]$ such that $l(R) \geq 4$. On the other hand, since D is nef, $-K_X \cdot \Gamma \leq 3L \cdot \Gamma$. Since $4 \leq -K_X \cdot \Gamma \leq 6$ (if $-K_X \cdot \Gamma = 7$, by [ChMiSB01], $X = \mathbb{P}^6$, contradicting $\rho = 2$), we have $L \cdot \Gamma \geq 2$ and $H \cdot \Gamma = 1$.
Let $R' := \mathbb{R}^+[\Gamma']$ be the other extremal ray of X.
We will call with Φ (resp. Φ') the contraction associated to R (resp. R') and with F (resp. F') an irreducible component of a (non trivial) fiber of the contraction Φ (resp. Φ').

Suppose first that R is of fiber type and R' is birational.
Then, by 1.3.17, both $\dim F$ and $\dim F'$ are ≥ 3. Since F and F' are (non trivial) fibers intersecting in one point, then $\dim F + \dim F' \leq 6$, which implies $\dim F = 3 = \dim F'$ for all fibers of Φ and Φ'.
Now, by 1.3.17, we find that $l(R) = 4$ and $l(R') = 3$.
Therefore Φ is an equidimensional fibration over a normal 3-fold W with general fiber $\mathbb{P}^3$; moreover, since $H \cdot \Gamma = 1$, by [BeSo95, Proposition 3.2.1], X has actually a

structure of $\mathbb{P}^3$-bundle onto a smooth 3-fold W.
By [AnOc02b, Theorem 5.2], Φ' is the blow-up of a smooth variety X' along a smooth subvariety of codimension 4; then $F' = \mathbb{P}^3$ which gives $W = \mathbb{P}^3$.
Put $\mathcal{E} := \Phi_* H$. Then $-K_X = 4H - \Phi^*(\mathcal{O}_{\mathbb{P}^3}(-4) + \det \mathcal{E})$. Therefore we find $\Phi^*(\det \mathcal{E} - 4\mathcal{O}_{\mathbb{P}^3}(1)) = 2H - L$, and $\Phi^* \mathcal{O}_{\mathbb{P}^3}(k-4) \cdot l' = 1$ for all lines l' contracted by Φ'. Then $k = 5$ and $\mathcal{E}$ has splitting type $(2, 1, 1, 1)$.
Therefore $\mathcal{E} = \mathcal{O}_{\mathbb{P}^3}(2) \oplus \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 3}$ and X is the blow-up of $\mathbb{P}^6$ along a $\mathbb{P}^2$.

Suppose now that both R and R' are of fiber type.
By 1.3.17, we find $\dim F' \geq 2$. If F and F' are general fibers intersecting in one point, then $\dim F + \dim F' \leq 6$, which implies $\dim F \leq 4$.
Moreover $l(R') = 3$, from which $L \cdot \Gamma' = 1 = H \cdot \Gamma'$.

Assume that $\dim F = 4$ and $\dim F' = 2$. Then both the contractions are equidimensional.
Since $l(R') = 3$ and $L \cdot \Gamma' = 1$, Φ' is a $\mathbb{P}^2$-bundle onto a smooth 4-fold Y, by [BeSo95, Proposition 3.2.1].
For the other ray, by 1.3.17, we have that $l(R) \leq 5$.
If $l(R) = 5$, since $H \cdot \Gamma = 1$, by [BeSo95, Proposition 3.2.1], X has a structure of $\mathbb{P}^4$-bundle onto a smooth surface W. Moreover, since $F' = \mathbb{P}^2$, we have that also $W = \mathbb{P}^2$.
By [Sa85, Theorem A], $X = \mathbb{P}^2 \times \mathbb{P}^4$, $L = \mathcal{O}(1,3)$ and $H = \mathcal{O}(1,1)$.
If $l(R) = 4$, since the contraction is equidimensional and $H \cdot \Gamma = 1$, then X is a quadric bundle onto a smooth surface. Moreover since $F' = \mathbb{P}^2$, we have that also $W = \mathbb{P}^2$.
Put $\mathcal{F} := \Phi'_* H$.
If $Y = \mathbb{Q}^4$, then $-K_X = 3H - \Phi'^*(\mathcal{O}_{\mathbb{Q}^4}(-4) + \det \mathcal{F})$. Therefore we find $\Phi'^*(\det \mathcal{F} - 4\mathcal{O}_{\mathbb{Q}^4}(1)) = H - L$, and $\Phi'^* \mathcal{O}_{\mathbb{Q}^4}(k-4) \cdot l = -1$ for all lines l contracted by Φ. Then $k = 3$ and $\mathcal{F}$ has splitting type $(1, 1, 1)$. Therefore $\mathcal{F} = \mathcal{O}_{\mathbb{Q}^4}(1)^{\oplus 3}$ and $X = \mathbb{P}^2 \times \mathbb{Q}^4$, $L = \mathcal{O}(1,2)$, $H = \mathcal{O}(1,1)$.
If $Y = \mathbb{P}^4$, then $-K_X = 3H - \Phi'^*(\mathcal{O}_{\mathbb{P}^4}(-5) + \det \mathcal{F})$. Therefore we find $\Phi'^*(\det \mathcal{F} - 5\mathcal{O}_{\mathbb{P}^4}(1)) = H - L$, and $\Phi'^* \mathcal{O}_{\mathbb{P}^4}(k-5) \cdot l = -1$ for all lines l contracted by Φ. Then $k = 4$ and $\mathcal{F}$ has splitting type $(2, 1, 1)$. Therefore $\mathcal{F} = \mathcal{O}_{\mathbb{P}^4}(2) \oplus \mathcal{O}_{\mathbb{P}^4}(1)^{\oplus 2}$ and X is the blow-up of $\mathbb{P}^6$ along a $\mathbb{P}^1$, which is a contradiction with the types of extremal rays of X.

Assume that $\dim F = 3 = \dim F'$. Then both the contractions are equidimensional.
As before, the contraction associated to R is a $\mathbb{P}^3$-bundle onto a smooth 3-fold W.
Since $l(R') = 3$, the contraction associated to R' is equidimensional and $H \cdot \Gamma' = 1$, X has a structure of quadric bundle over a normal 3-fold Y.
Therefore, since $F' = \mathbb{Q}^3$ and $F = \mathbb{P}^3$, we find $Y = \mathbb{P}^3$ and either $W = \mathbb{P}^3$ or $W = \mathbb{Q}^3$.
Put $\mathcal{F} := \Phi_* H$.
If $W = \mathbb{Q}^3$, then $-K_X = 4H - \Phi^*(\mathcal{O}_{\mathbb{Q}^3}(-3) + \det \mathcal{F})$. Therefore we find $\Phi^*(\det \mathcal{F} -$

$3\mathcal{O}_{\mathbb{Q}^3}(1)) = 2H - L$, and $\Phi^*\mathcal{O}_{\mathbb{Q}^3}(k-3)\cdot l = 1$ for all lines l contracted by Φ'. Then $k = 4$ and $\mathcal{F}$ has splitting type $(1,1,1,1)$. Therefore $\mathcal{F} = \mathcal{O}_{\mathbb{Q}^3}(1)^{\oplus 4}$ and $X = \mathbb{P}^3 \times \mathbb{Q}^3$, $L = \mathcal{O}(2,1)$, $H = \mathcal{O}(1,1)$.
If $W = \mathbb{P}^3$, then $-K_X = 4H - \Phi^*(\mathcal{O}_{\mathbb{P}^3}(-4) + \det \mathcal{F})$. Therefore we find $\Phi^*(\det \mathcal{F} - 4\mathcal{O}_{\mathbb{P}^3}(1)) = 2H - L$, and $\Phi^*\mathcal{O}_{\mathbb{P}^3}(k-4)\cdot l = 1$ for all lines l contracted by Φ'. Then $k = 5$ and $\mathcal{F}$ has splitting type $(2,1,1,1)$. Therefore $\mathcal{F} = \mathcal{O}_{\mathbb{P}^3}(2) \oplus \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 3}$ and X is the blow-up of $\mathbb{P}^6$ along a $\mathbb{P}^2$, which is a contradiction with the types of extremal rays of X.

Assume that $\dim F = 3$, $\dim F' = 2$ and both the contractions are equidimensional.
As before, the contraction associated to R is a $\mathbb{P}^3$-bundle onto a smooth 3-fold W.
As before, the contraction associated to R' is a $\mathbb{P}^2$-bundle onto a smooth 4-fold Y.
Therefore, by [OcWi02, Theorem 2], W and Y have structures of $\mathbb{P}$-bundle onto a smooth curve C and $X \times_C W$. But a Fano manifold X of pseudoindex i_X cannot have an extremal contraction onto varieties of dimension $\leq i_X - 2$.

Assume that $\dim F = 3$, $\dim F' = 2$ and there exists a fiber of Φ', namely $\widetilde{F'}$, of dimension at least 3.
Since $\dim F + \dim \widetilde{F'} \leq 6$, we find that $\dim \widetilde{F'} = 3$ and Φ is equidimensional.
As before, the contraction associated to R is a $\mathbb{P}^3$-bundle onto a smooth 3-fold W.
The contraction associated to R' is a scroll onto a normal 4-fold Y.

Assume that $\dim F = 3$, $\dim F' = 2$ and there exists a fiber of Φ, namely $\widetilde{F}$, of dimension at least 4.
Since $\dim \widetilde{F} + \dim F' \leq 6$, we find that $\dim \widetilde{F} = 4$ and Φ' is equidimensional.
As before, the contraction associated to R' is a $\mathbb{P}^2$-bundle onto a smooth 4-fold W.
The contraction associated to R is a scroll onto a normal 3-fold Y.

Suppose now that R is birational and R' is of fiber type.
By 1.3.17 we find that $5 \geq \dim F \geq l(R) \geq 4$.
If $\dim F = 5$, then the contraction associated to R contracts a divisor to a point, contradicting the types of extremal rays of X in [ChOcpr, Theorem 1.1].
Therefore $\dim F = 4 = l(R)$. By [AnOc02b, Theorem 5.2], X is the blow-up of a smooth variety X' along a smooth subvariety of codimension 5.
Since $\dim F' = 2$ and $H \cdot \Gamma' = 1$, R' gives to X a structure of $\mathbb{P}^2$-bundle onto a smooth 4-fold W, by [BeSo95, Proposition 3.2.1]. Moreover, since $F = \mathbb{P}^4$, we have that also $W = \mathbb{P}^4$.
Put $\mathcal{G} := \Phi_* H$.
If $W = \mathbb{P}^4$, then $-K_X = 3H - \Phi'^*(\mathcal{O}_{\mathbb{P}^4}(-5) + \det \mathcal{F})$. Therefore we find $\Phi'^*(\det \mathcal{F} - 5\mathcal{O}_{\mathbb{P}^4}(1)) = H - L$, and $\Phi'^*\mathcal{O}_{\mathbb{P}^4}(k-5)\cdot l = -1$ for all lines l contracted by Φ. Then $k = 4$ and $\mathcal{F}$ has splitting type $(2,1,1)$. Therefore $\mathcal{F} = \mathcal{O}_{\mathbb{P}^4}(2) \oplus \mathcal{O}_{\mathbb{P}^4}(1)^{\oplus 2}$ and X is the blow-up of $\mathbb{P}^6$ along a $\mathbb{P}^1$. *q.e.d.*

Now, we note that part of the proofs of 3.4.4 and 3.4.5 can be written in a general way. Therefore we give the following result:

Proposition 3.4.6. *Let X, L and Z be as in 3.4.1. Assume that* $\dim Z \geq i+1$, *with* $i \in \mathbb{Z}_{\geq 3}$, *and that Z is a Fano manifold of index $n-i$, i.e. $-K_Z = (n-i)H_Z$ for some ample line bundle H_Z on Z. Assume that the Picard number of Z is greater than* 1.
Then $\dim X \leq 2i-1$, *unless $X = \mathbb{P}^i \times \mathbb{P}^i$ and $L = H = \mathcal{O}(1,1)$, i.e. $Z = \mathbb{P}(T_{\mathbb{P}^i})$.*

Proof. Since $\rho := \rho(X) = \rho(Z) > 1$ and $\operatorname{index}(Z) \leq 2$, by [Wi90a, Theorem A] applied to Z, we have that

$$n - i \leq \frac{\dim Z + 2}{2} = \frac{n+1}{2}, \quad \text{i.e. } n \leq 2i+1.$$

Assume that $n = 2i+1$. By [Wi90a, Theorem B], $Z = \mathbb{P}^i \times \mathbb{P}^i$, that cannot be an ample section of any projective manifold, (see [BeSo95, Corollary 5.2.4]). Then we have that $n \leq 2i$.
If $n = 2i$ then, by [Wi91b], we have the following possibilities for Z:

1. $Z = \mathbb{P}^{i-1} \times \mathbb{Q}^i$.

2. $Z = \mathbb{P}(T_{\mathbb{P}^i})$, $H_Z = \mathcal{O}(1,1)$.

3. $Z = \mathbb{P}_{\mathbb{P}^i}(\mathcal{O}_{\mathbb{P}^i}(2) \oplus \mathcal{O}_{\mathbb{P}^i}(1)^{\oplus(i-1)})$.

Only $Z = \mathbb{P}(T_{\mathbb{P}^i})$ is an ample divisor in a smooth X. In fact, the first case is ruled out by [BeSo95, Corollary 5.2.4] and the third case is ruled out by [Fu80, Theorem 5.5].
Note that H_Z is spanned so we can apply theorem 3.4.2 which gives that H is ample. By [AnOc99, Theorem 4.1] each contraction on $Z = \mathbb{P}(T_{\mathbb{P}^i})$ lifts to a $\mathbb{P}^i$-bundle on X. It is straightforward to prove now that $X = \mathbb{P}^i \times \mathbb{P}^i$ and that $L = H = \mathcal{O}(1,1)$. *q.e.d.*

3.4.1 Ample sections polarized by vector bundles

Now, it seems natural to generalize the results of the section by investigating the manifolds X whose hyperplane section, Z, has anticanonical bundle $-K_Z = \det \mathcal{V}$, for some ample vector bundle $\mathcal{V}$ of rank r on Z.

Remark 3.4.7. If X, L and Z are as in 3.4.1, $\dim Z \geq 2$ and $-K_Z = \det \mathcal{V}$ for some ample vector bundle $\mathcal{V}$ on Z of rank r, then K_X is not nef.
Indeed, if Γ a rational curve of Z, we have $-(K_X + L) \cdot \Gamma = -K_Z \cdot \Gamma = \det \mathcal{V} \cdot \Gamma \geq r$; then $K_X \cdot \Gamma \leq -L \cdot \Gamma - r < 0$.

Remark 3.4.8. Note that, even being $\mathcal{V}$ ample, it is not true in general that $\mathcal{V}$ is the restriction of a vector bundle $\mathcal{F}$ on X. Indeed, take $\mathcal{F}$ a sheaf on $X \subset \mathbb{P}^n$ which is not locally free at a point $x \in X$. Take Z a hyperplane section such that $x \notin Z$.

As a first step we give the following characterization of the projective space, which was proved in a different way by T. Peternell in [Pe90, Theorem].

Proposition 3.4.9. *Let X be a smooth complex projective variety of dimension n and let $\mathcal{E}$ be an ample vector bundle of rank $r \geq n+1$ on X such that $-K_X = \det \mathcal{E}$. Then $r = n+1$ and $(X, \mathcal{E}) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n+1)})$.*

Proof. If $R := \mathbb{R}^+[C]$ is an extremal ray in the face supported by $K_X + \det \mathcal{E}$, then $l(R) = n+1 = r$. Therefore $1 = \tau(X, \det \mathcal{E}) = \frac{n+1}{r}$. Then the assertion follows by 2.2.1. *q.e.d.*

Proposition 3.4.10. *Let X, L and Z be as in 3.4.1. Assume that* $\dim Z \geq 3$ *and* $-K_Z = \det \mathcal{V}$ *for some ample vector bundle $\mathcal{V}$ on Z of rank $r \geq n-1$.*
Then one of the following holds:

1. *$r = n$ and $(X, L) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$.*
2. *$r = n-1$ and (X, L) is one of the following:*
   - *(a) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(i))$, $i = 1, 2$.*
   - *(b) $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$.*

Proof. If $r \geq n$, then $(Z, \mathcal{V}) = (\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1)^{\oplus n})$, by 3.4.9. Therefore $(X, L) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$.
Assume now that $r = n-1$. If $R := \mathbb{R}^+[C]$ is an extremal ray in the face supported by $K_Z + \det \mathcal{V}$, then $l(R) \geq n-1$.
If $l(R) = n$, then, by [ChMiSB01], $Z = \mathbb{P}^{n-1}$ and $(X, L) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$.
If $l(R) = n-1 = r$, then $1 = \tau(Z, \det \mathcal{V}) = \frac{n-1}{r}$; therefore, by 2.3.1, $(Z, \mathcal{V}) = (\mathbb{Q}^{n-1}, \mathcal{O}_{\mathbb{Q}^{n-1}}(1)^{\oplus r})$. Therefore (X, L) is either $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2))$ or $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$. *q.e.d.*

Theorem 3.4.11. *Let X, L and Z be as in 3.4.1. Assume that* $\dim Z \geq 3$ *and* $-K_Z = \det \mathcal{V}$ *for some ample vector bundle $\mathcal{V}$ on Z of rank $r = n-2$.*
Then one of the following holds:

1. *$(X, L) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(i))$, $i = 1, 2, 3$.*

2. $(X,L) = (\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(j))$, $j = 1,2$.

3. (X,L) *is a del Pezzo manifold,* $\rho(X) = 1$, $L = \mathcal{O}_X(1)$ *is the ample generator of* $\mathrm{Pic}(X)$.

4. $\dim X = 4$ *and*

   (a) $X = \mathbb{P}^2 \times \mathbb{P}^2$.

   (b) X *is a* $\mathbb{P}^2$*-bundle over* $\mathbb{Q}^2$

   (c) X *is a* $\mathbb{P}^3$*-bundle over* $\mathbb{P}^1$.

   (d) X *is a quadric bundle over* $\mathbb{P}^1$

Proof. Assume $\rho := \rho(X) = \rho(Z) = 1$.
Let R be an extremal ray of Z and let Γ_Z be a minimal extremal curve in R. Then the length of the ray is $l(R) = -K_Z \cdot \Gamma_Z = \det \mathcal{V} \cdot \Gamma_Z \geq r = n-2$.
If $l(R) = n$, then $Z \cong \mathbb{P}^{n-1}$, by [ChMiSB01]; hence $(X,L) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$.
If $l(R) = n-1$, by [PeSzWi92] and [Ocpr2], $(Z,\mathcal{V})$ is one of the following:

1. $(\mathbb{Q}^{n-1}, \mathcal{O}_{\mathbb{Q}^{n-1}}(2) \oplus \mathcal{O}_{\mathbb{Q}^{n-1}}(1)^{\oplus(n-3)})$.
2. $(\mathbb{Q}^4, \mathbb{E}(2) \oplus \mathcal{O}_{\mathbb{Q}^4}(1))$, where $\mathbb{E}$ is the spinor bundle over $\mathbb{Q}^4$.
3. $(\mathbb{Q}^3, \mathbb{E}(2))$, where $\mathbb{E}$ is the spinor bundle over $\mathbb{Q}^3$.

In all the three cases (X,L) is either $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2))$, or $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$.
If $l(R) = n-2$, then $1 = \tau(Z, \det \mathcal{V}) = \frac{n-2}{r}$; therefore, by 2.4.1, $(Z, \mathcal{L})$ is a del Pezzo manifold and $\mathcal{V} = \mathcal{L}^{\oplus r}$. Therefore, by [AnOc99, Corollary 6.5], (X,L) is either $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(3))$, or $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(2))$, or (X,L) is a del Pezzo manifold and $L \cong \mathcal{O}_X(1)$ is the ample generator of $\mathrm{Pic}(X)$.
We can thus assume $\rho := \rho(X) = \rho(Z) > 1$.
By [Wi90a, Theorem A] applied to Z, we have that

$$n - 2 \leq \frac{\dim Z + 2}{2} = \frac{n+1}{2}$$

with equality if and only if $n = 5$.
If $n = 5$, by [AnMe97, Lemma 5.3], we have $Z = \mathbb{P}^2 \times \mathbb{P}^2$.
Since, by [BeSo95, Corollary 5.2.4], this cannot be an ample section of any projective manifold, we have only $n = 4$. By [PeSzWi92, Theorem 0.4], we have the following possibilities for $(Z, \mathcal{V})$:

1. $(\mathbb{P}^2 \times \mathbb{P}^1, pr^* T_{\mathbb{P}^2} \otimes \mathcal{O}(0,1)$ or $\mathcal{O}(2,1) \oplus \mathcal{O}(1,1))$.

2. $(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1,1,1)^{\oplus 2})$.

3. Z is a del Pezzo manifold and $\mathcal{V} \cong \mathcal{O}(1)^{\oplus 2}$.

By [BeSo95, Corollary 5.2.4], the second case cannot be an ample section in any projective manifold.
If $Z = \mathbb{P}^2 \times \mathbb{P}^1$, by [AnOc99, Proposition 4.9 and Remark 4.11], X is a $\mathbb{P}^3$-bundle over $\mathbb{P}^1$.
In the third case, Z is one of the following:

1. $Z = \mathbb{P}(T_{\mathbb{P}^2})$;

2. $Z = \mathbb{P}^1 \times \mathbb{Q}^2$;

3. $Z = Bl_P\mathbb{P}^3$.

In the first case, by [AnOc99, Theorem 4.1], each contraction on Z lifts to a $\mathbb{P}^2$-bundle on X. Therefore, by [Sa85, Theorem A], $X = \mathbb{P}^2 \times \mathbb{P}^2$.
In the second case, by [AnOc99], X is either a $\mathbb{P}^2$-bundle onto $\mathbb{Q}^2$ or a quadric bundle onto $\mathbb{P}^1$.
In the last case, Z cannot be an ample section in any manifold by [Fu80, Theorem 5.5]. *q.e.d.*

Theorem 3.4.12. *Let* X, L *and* Z *be as in 3.4.1. Assume that* $\dim Z \geq 4$, $\rho(Z) > 1$ *and* $-K_Z = \det \mathcal{V}$ *for some ample vector bundle* $\mathcal{V}$ *on* Z *of rank* $r = n - 3$.
Then one of the following holds:

1. $\dim X = 6$ *and* $X = \mathbb{P}^3 \times \mathbb{P}^3$, $L = \mathcal{O}(1,1)$, *i.e.* $Z = \mathbb{P}(T_{\mathbb{P}^3})$.

2. $\dim X = 5$ *and*

   (a) X *is a projective bundle over* $\mathbb{P}^1$ *and* $Z = \mathbb{P}^1 \times \mathbb{P}^3$.

   (b) X *is a del Pezzo fibration over* $\mathbb{P}^1$ *and* $Z = \mathbb{P}^1 \times V$, *with* V *a threefold of index* 2 *and of Picard number one.*

   (c) $X = \mathbb{P}^2 \times \mathbb{P}^3$, $L = \mathcal{O}(1,2)$.

   (d) X *is a Mukai 5-fold, i.e.* $-K_X = 3L$, *(with* $\rho = 2$*). According to [Wi91b] they are:*

      i. $X = \mathbb{P}^2 \times \mathbb{Q}^3$, $L = \mathcal{O}(1,1)$.

      ii. $X = \mathbb{P}(T_{\mathbb{P}^3})$, $L = \mathcal{O}(1,1)$.

iii. $X = \mathbb{P}_{\mathbb{P}^3}(\mathcal{O}_{\mathbb{P}^3}(2) \oplus \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 2})$, $L = \xi + p^*\mathcal{O}_{\mathbb{P}^3}(1)$.

(e) $\mathrm{index}(Z) = 1$, $\mathrm{pseudoindex}(Z) = 2$ *and* $2 \le \rho \le 3$ *(therefore* $\mathcal{V} \neq \mathcal{L}^{\oplus 2}$ *for all* $\mathcal{L}$ *ample line bundle on* Z*).*

Proof.
Since $\rho := \rho(X) = \rho(Z) > 1$, by [Wi90a, Theorem A] applied to Z, we have that

$$n - 3 \le \frac{\dim Z + 2}{2} = \frac{n+1}{2}, \quad \text{i.e. } n \le 7.$$

If $n = 7$, by [AnMe97, Lemma 5.3], we have $Z = \mathbb{P}^3 \times \mathbb{P}^3$.
Since, by [BeSo95, Corollary 5.2.4], this cannot be an ample section of any projective manifold, we have that $n \le 6$.
If $n = 6$, then, by [Oc01, Theorem 1], we have the following possibilities for $(Z, \mathcal{V})$:

1. $(\mathbb{P}^2 \times \mathbb{Q}^3, \mathcal{O}(1,1)^{\oplus 3})$.
2. $(\mathbb{P}^2 \times \mathbb{P}^3, \mathcal{O}(1,2) \oplus \mathcal{O}(1,1)^{\oplus 2}$ or $\mathcal{O}(1,0) \otimes p_2^* T_{\mathbb{P}^3})$.
3. $(\mathbb{P}(T_{\mathbb{P}^3}), \mathcal{O}(1,1)^{\oplus 3})$.
4. $(\mathbb{P}_{\mathbb{P}^3}(\mathcal{O}_{\mathbb{P}^3}(2) \oplus \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 2}), \mathcal{O}(1,1)^{\oplus 3})$.

Only $Z = \mathbb{P}(T_{\mathbb{P}^3})$ is an ample divisor in a smooth X. In fact, the first two cases are ruled out by [BeSo95, Corollary 5.2.4] and the last case is ruled out by [Fu80, Theorem 5.8].
Note that $-K_Z = 3A_Z$, where $A_Z = \mathcal{O}(1,1)$ is spanned; so we can apply theorem 3.4.2 which gives that A is ample. By [AnOc99, Theorem 4.1] each contraction on $Z = \mathbb{P}(T_{\mathbb{P}^3})$ lifts to a $\mathbb{P}^3$-bundle on X. It is straightforward to prove now that $X = \mathbb{P}^3 \times \mathbb{P}^3$ and that $L = A = \mathcal{O}(1,1)$.
If $n = 5$, observe that $1 \le \mathrm{index}(Z) \le 3$ (otherwise $\rho = 1$).
If $\mathrm{index}(Z) = 3$, by [Wi90a, Theorem B], $Z \cong \mathbb{P}^2 \times \mathbb{P}^2$, that, by [BeSo95, Corollary 5.2.4], cannot be an ample section of any projective manifold.
If $\mathrm{index}(Z) = 2$, we are in the hypothesis of Theorem 3.4.4, and we find the cases 2a, 2b, 2c, 2d.
Assume now that $\mathrm{index}(Z) = 1$.
By [BoCaDeDr], $\rho \cdot (i_Z - 1) \le 4$, with equality if and only if $Z = (\mathbb{P}^{i_Z - 1})^\rho$, where i_Z is the pseudoindex of Z. Then either $\rho = 2$ and $2 \le i_Z \le 3$, or $\rho = 3$ and $i_Z = 2$, or $\rho = 4$ and $i_Z = 2$.
In the last case $Z = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, whose index is not 1. Moreover if $\rho = 2$ and $i_Z = 3$, then $Z = \mathbb{P}^2 \times \mathbb{P}^2$, whose index is not 1.

q.e.d.

Remark 3.4.13. In case L is assumed to be very ample, the case $Z = \mathbb{P}^1 \times V$, with (V, A) a del Pezzo threefold (with respect to some ample line bundle A) of Picard number one, can be excluded in theorems 3.4.3, 3.4.4 and 3.4.12, by [BeFaSopr, Proposition 0.1].

Chapter 4

Terminal Veronese Fibrations

Let X be a projective variety of dimension n with at most terminal $\mathbb{Q}$-factorial singularities.
One of the main conjectures of Mory theory states that there exists a minimal model X' birational to X if and only if X is not uniruled. This conjecture has been proved in the 3-dimensional case. In particular, via the MMP, it is possible to associate a Mori fiber space (4.1.1) to a uniruled 3-fold.
However, even though the MMP for 3-folds has been solved, it is difficult to follow the birational modifications that occur along the program. But, in the special case of 3-folds with a polarizing divisor H with small H^3, the program (called in this case $\sharp$-*MMP*) can be explicitely developed: a list (4.1.9) of $\sharp$-minimal models of pairs $(X, \mathcal{H})$, given by a terminal $\mathbb{Q}$-factorial uniruled 3-fold X and a big uniruled system (4.1.7) $\mathcal{H}$, has been given by M. Mella in [Me02].
In this chapter, which is part of a work in progress in collaboration with A. Sarti and A.L. Knutsen ([KnNoSa]), we deal with some Mori fiber spaces. In particular, we consider $(X, \mathcal{H})$, a pair consisting of an irreducible terminal $\mathbb{Q}$-factorial 3-fold X and a big line bundle $\mathcal{H}$ on X such that the linear system $|\mathcal{H}|$ is base point free.
We give the definition of *(3-dimensional) terminal Veronese fibration* (4.2.1) and we study surfaces S in $|\mathcal{H}|$ under the assumptions that $S \cap \mathrm{Sing}X = \emptyset$ and that the general fiber of $S \longrightarrow B$ is a smooth quartic (2.1.2). Since any $S \in |\mathcal{H}|$ is fibered over the smooth curve B, we study all possible kinds of fiber.

4.1 Mori Fiber Spaces and Uniruled Varieties

We recall here the definition of "Mori fiber space" and "uniruled variety" and the relation between the two objects.

Definition 4.1.1. A Mori fiber space is a Fano-Mori contraction $\pi \colon X \longrightarrow Y$ of a terminal $\mathbb{Q}$-factorial variety X such that $\dim Y < \dim X$ and $\operatorname{rk} \operatorname{Pic}(X/Y) = 1$.

Remark 4.1.2. In general the Mori fiber space associated to a variety by the Minimal Model Program is not uniquely determined (the problem arises when two extremal rays have non disjoint exceptional loci).

Definition 4.1.3. A variety X is *uniruled* if there exists a generically finite surjective map $Y \times \mathbb{P}^1 \longrightarrow X$.

Proposition 4.1.4. [KaMaMa87, Corollary 5.1.4] *Let $\pi \colon X \longrightarrow Y$ be a Mori fiber space. Then X is a uniruled variety.*

In general, the converse of the previous result is still an open problem. Indeed the minimal model conjecture states that a projective variety X with at most terminal $\mathbb{Q}$-factorial singularities has a minimal model X' birational to X if and only if X is not uniruled.
Now, in the 3-dimensional case, the conjecture has been proved; thus, via MMP, we can associate a Mori fiber space to a uniruled 3-fold X. The main difficulty, in order to study the birational geometry of X, is that the birational modifications occurring along the MMP are difficult to follow and usually it is almost impossible to guess what is the output. However, under strong assumptions on the variety and the use of a polarizing divisor, it is possible to understand the output.
This was done by M. Mella in [Me02]. In the following we recall the definitions and the main result given [Me02].

4.1.1 $\sharp$-minimal model

In this section we recall the definition of $\sharp$-*minimal model* of a pair $(X, \mathcal{H})$.

Definition 4.1.5. [Me02, Section 2, Definition 3.1 and Remark 3.5] Let X be a terminal $\mathbb{Q}$-factorial uniruled 3-fold and let $\mathcal{H}$ a *movable* linear system, i.e. $\dim |mH| > 0$ for $m \gg 0$, with general element $H \in \mathcal{H}$ on X. Assume that H is nef. Then the *threshold* of the pair $(X, \mathcal{H})$ is defined as

$$\rho_{\mathcal{H}} = \rho(X, \mathcal{H}) := \sup \left\{ m \in \mathbb{Q} \;\middle|\; H + mK_X \text{ is an effective } \mathbb{Q}\text{-divisor} \right\}.$$

Since we are assuming that $\dim \mathcal{H} \geq 0$, we have that $\rho \geq 0$.
In particular the threshold is invariant under birational transformations.

Definition 4.1.6. [Me02, Definition 3.3] A pair $(X^\sharp, \mathcal{H}^\sharp)$ is called a *$\sharp$-minimal model* of $(X, \mathcal{H})$ if:

1. $X^\sharp$ has a Mori fiber space structure $\pi\colon X^\sharp \longrightarrow W$ and $\mathcal{H}^\sharp$ is a movable linear system of Weil divisors;

2. there exists a birational map $\psi\colon X \dashrightarrow X^\sharp$ such that $\mathcal{H}^\sharp = \psi_* \mathcal{H}$;

3. if $H^\sharp \in \mathcal{H}^\sharp$ is a general member, then $\rho(X, \mathcal{H}) K_{X^\sharp} + H^\sharp \equiv_\pi \mathcal{O}_{X^\sharp}$, where $\equiv_\pi$ denotes the numerical equivalence on curves contracted by π.

4.1.2 3-folds with a big uniruled system

In this section we give the main theorem of [Me02], together with some remarks on that result. We will need first the following definition.

Definition 4.1.7. [Me02, Definition 5.1] Let X be a terminal $\mathbb{Q}$-factorial 3-fold and $\mathcal{H}$ a movable linear system. We say that $(X, \mathcal{H})$ is a *pair with a big uniruled system* if $H \in \mathcal{H}$ is nef and big and H is a smooth surface of negative Kodaira dimension.

Proposition 4.1.8. *[Me02, Lemma 5.2] Let $(X, \mathcal{H})$ be a pair with a big uniruled system. Then X is uniruled and $\rho(X, \mathcal{H}) < 1$.*

Theorem 4.1.9. *[Me02, Theorem 5.3] Let $(X, \mathcal{H})$ be a pair with big uniruled system. Then $(X^\sharp, \mathcal{H}^\sharp)$ is one of the following:*

(I) a $\mathbb{Q}$-Fano 3-fold of index $1/\rho > 1$, with $K_{X^\sharp} \sim -1/\rho H^\sharp$, belonging to the following list:

- *(i)* $(\mathbb{P}(1,1,2,3), \mathcal{O}(6))$*;*
- *(ii)* $(X_6 \subset \mathbb{P}(1,1,2,3,a), X_6 \cap \{x_4 = 0\})$*, with* $3 \leq a \leq 5$*;*
- *(iii)* $(X_6 \subset \mathbb{P}(1,1,2,2,3), X_6 \cap \{x_3 = 0\})$*;*
- *(iv)* $(X_6 \subset \mathbb{P}(1,1,1,2,3), X_6 \cap \{x_0 = 0\})$*;*
- *(v)* $(\mathbb{P}(1,1,1,2), \mathcal{O}(4))$*;*
- *(vi)* $(X_4 \subset \mathbb{P}(1,1,1,1,2), X_4 \cap \{x_0 = 0\})$*;*
- *(vii)* $(X_4 \subset \mathbb{P}(1,1,1,2,a), X_4 \cap \{x_4 = 0\})$*, with* $2 \leq a \leq 3$*;*
- *(viii)* $(\mathbb{P}^3, \mathcal{O}(a))$*, with* $a \leq 3$*;*

(ix) $(\mathbb{Q}^3, \mathcal{O}(b))$, *with* $b \leq 2$;

(x) $(X_3 \subset \mathbb{P}(1,1,1,1,2), X_3 \cap \{x_4 = 0\})$;

(xi) $(X_3 \subset \mathbb{P}^4, \mathcal{O}(1))$;

(xii) $(X_{2,2} \subset \mathbb{P}^5, \mathcal{O}(1))$;

(xiii) *a linear section of* $\mathrm{Grass}(1,4) \subseteq \mathbb{P}^9$;

(xiv) $(\mathbb{P}(1,1,1,2), \mathcal{O}(2))$, *the cone over the Veronese surface.*

(II) *a fibration over a smooth curve with generic fiber* $(F, H^\sharp_{|F}) \cong (\mathbb{P}^2, \mathcal{O}(2))$ *and with at most finitely many fibers* $(G, H^\sharp_{|G}) \cong (\mathbf{S_4}, \mathcal{O}(\mathbf{1}))$, *where* $\mathbf{S_4}$ *is the cone over the normal quartic curve and the vertex sits over a hyperquotient singularity of type* $1/2(1,-1,1)$ *with* $f = xy - z^2 + t^k$, *for* $k \geq 1$.

(III) *a quadric bundle with at most* cA_1 *singularities of type* $f = x^2 + y^2 + z^2 + t^k$, *for* $k \geq 2$, *and* $H^\sharp_{|F} \sim \mathcal{O}(1)$.

(IV) $(\mathbb{P}(E), \mathcal{O}(1))$ *where* E *is a rank* 3 *vector bundle over a smooth curve.*

(V) $(\mathbb{P}(E), \mathcal{O}(1))$ *where* E *is a rank* 2 *vector bundle over a surface of negative Kodaira dimension.*

Remark 4.1.10. We observe that, in the list of $\mathbb{Q}$-Fano 3-folds in (I) of 4.1.9, the map $\varphi_{\mathcal{H}^\sharp}$ defined by the system $|\mathcal{H}^\sharp|$ is not birational in the cases (iii), (iv) and (vi). Moreover in the cases (II)-(V) M. Mella gives the exact value of the threeshold ρ is respectively equal to 2/3, 1/2, 1/3 and 1/2.

Now set $d^\sharp := (\mathcal{H}^\sharp)^3$ and $n^\sharp := h^0(\mathcal{H}^\sharp) - 1$. It is quite easy to compute $d^\sharp$ and $n^\sharp$ in the case (I) above. We explain it making the computations of case (ii) as an example. Consider $(X_6 \in \mathbb{P}(1,1,2,3,a), X_6 \cap \{x_4 = 0\})$, with $3 \leq a \leq 5$.
Write $A = \mathcal{O}(1)$. Then $A^3 = 6/1 \cdot 1 \cdot 2 \cdot 3 \cdot a = 1/a$ by the standard formula for complete intersection. Now, the polarisation is given by aA (because x_4 has degree a), so we get $(aA)^3 = a^2$. On the other hand, $h^0(aA) =$ number of monomials of degree a in $k[x_0, x_1, x_2, x_3, x_4]$.
Arguing in this way, we obtain the following table.

Case		$d^\sharp$	$n^\sharp$	Case		$d^\sharp$	$n^\sharp$
(i)		36	22	(ii)	$a=3$	9	7
(ii)	$a=4$	16	11	(ii)	$a=5$	25	16
(iii)		4	4	(iv)		1	2
(v)		32	21	(vi)		2	3
(vii)	$a=2$	8	7	(vii)	$a=3$	18	13
(viii)	$a=1$	1	3	(viii)	$a=2$	8	9
(viii)	$a=3$	27	19	(ix)	$b=1$	2	4
(ix)	$b=2$	16	14	(x)		12	10
(xi)		3	4	(xii)		4	5
(xiii)		5	6	(xiv)		4	6

In the study of uniruled 3-folds, we will need the following general definition:

Definition 4.1.11. (cf. [Ko96, IV 1.4]) Let X be an n-dimensional projective variety, H a nef $\mathbb{Q}$-divisor on X and $d \in \mathbb{Q}$. We say that X is *uniruled of H-degree* at most d if there is a dense open subset $U \subset X$ such that every point in U is contained in a rational curve C with $C \cdot H \leq d$.
In particular, if $X \subset \mathbb{P}^n$, we say that X is *uniruled by lines* if $d = 1$ with respect to $H := \mathcal{O}_X(1)$.

4.2 Terminal Veronese Fibrations

We start this section with the following definition:

Definition 4.2.1. Let $(X, \mathcal{H})$ be a pair satisfying the following:

1. X is a 3-dimensional terminal irreducible variety with a Mori fiber space structure $p\colon X \longrightarrow B$, where B is a smooth curve.

2. $\mathcal{H}$ is a line bundle on X such that the system $|\mathcal{H}|$ contains a smooth surface and $\mathcal{H}^3 > 0$.

3. The general fiber of p is $(F, \mathcal{H}_{|F}) \cong (\mathbb{P}^2, \mathcal{O}(2))$ and the rest are at most finitely many fibers $(G, \mathcal{H}_{|G}) \cong (\mathbf{S_4}, \mathcal{O}(\mathbf{1}))$, where $\mathbf{S_4}$ is the cone over the normal quartic curve.

Such a Mori fiber space will be called a *(3-dimensional) terminal Veronese fibration.*

In particular, the 3-folds in point (II) of theorem 4.1.9 are terminal Veronese fibrations. Indeed, by [Me02, Corollary 3.10 and Lemma 5.2], $\mathcal{H} \in \mathrm{PicX}$, $|\mathcal{H}|$ has at most finitely many base points and the general element in $|\mathcal{H}|$ is smooth and irreducible. The easiest examples of terminal Veronese fibrations are the smooth ones as in the following

Example 4.2.2. Consider $X = \mathbb{P}^2 \times \mathbb{P}^1$ with projections p and q respectively and let $\mathcal{H} := p^*\mathcal{O}_{\mathbb{P}^2}(2) \otimes q^*\mathcal{O}_{\mathbb{P}^1}(a)$ for $a \in \mathbb{N}_{\geq 1}$.
Note that $n := h^0(\mathcal{H}) - 1 = h^0(\mathcal{O}_{\mathbb{P}^2}(2)) \cdot h^0(\mathcal{O}_{\mathbb{P}^1}(a)) - 1 = 6(a+1) - 1$ and $d := \mathcal{H}^3 = (p^*\mathcal{O}_{\mathbb{P}^2}(2) \otimes q^*\mathcal{O}_{\mathbb{P}^1}(a))^3 = 3(p^*\mathcal{O}_{\mathbb{P}^2}(2))^2 \cdot q^*\mathcal{O}_{\mathbb{P}^1}(a) = 12a$, whence $d = 2n - 10$.

But there are also singular such varieties: take $\mathbb{P}^2 \times \mathbb{P}^1$ and blow up a conic C in a fiber and contract the strict transform of C, thus producing a Veronese cone singularity.

4.2.1 Sections of Terminal Veronese Fibrations

In this section we will study "hyperplane sections" of X, i.e. surfaces in $|\mathcal{H}|$.
Now we fix a terminal Veronese fibration $(X, \mathcal{H})$.
We want to see how the fibers of $(X, \mathcal{H})$ looks like. We denote by $\mathcal{V} \in \mathrm{Num}X$ the numerical equivalence class of a fiber. We have

$$\mathcal{H}^2 \cdot \mathcal{V} = 4, \quad \mathcal{H} \cdot \mathcal{V}^2 = \mathcal{V}^3 = 0. \tag{2.1.1}$$

Now any element $S \in |\mathcal{H}|$ such that $S \cap \mathrm{Sing}X = \emptyset$ is a surface (not necessarily reduced or irreducible) fibered over B, with fibers being either a whole Veronese surface, or a smooth quartic, or a union of two distinct conics intersecting in one point or a double conic.
Assume now that S has one fiber which is a smooth quartic. Then the general fiber is a smooth quartic. In other words we make from now on the assumptions:

$$\begin{gathered} S \in |\mathcal{H}| \text{ such that } S \cap \mathrm{Sing}X = \emptyset \text{ and} \\ \text{the general fiber of } S \longrightarrow B \text{ is a smooth quartic.} \end{gathered} \tag{2.1.2}$$

Then we can write, in $\mathrm{Num}X$,

$$S \equiv S_0 + m\mathcal{V}, \quad m \in \mathbb{Z}_{\geq 0}, \tag{2.1.3}$$

where S_0 is a reduced and irreducible surface (and Cartier in X) fibered over B.
By [Me02] X is normal and has only terminal singularities, so it has only rational singularities by [KoMo, Theorem 5.22] whence it is Cohen-Macaulay by [KoMo, Theorem 5.10]. It follows that S_0 is also Cohen-Macaulay (being a local complete intersection and lying outside $\mathrm{Sing}X$). Now X is smooth in codimension 2 (since

it is terminal) and $\omega_X = \mathcal{O}_X(K_X)$ by [KoMo, Proposition 5.75]. Moreover K_X is $\mathbb{Q}$-Cartier and even Cartier off Sing$X$. So by adjunction (cf. [KoMo, Proposition 5.73])

$$\omega_{S_0} \cong \omega_X(S_0) \otimes \mathcal{O}_{S_0} \cong \mathcal{O}_{S_0}(K_X + S_0). \qquad (2.1.4)$$

In particular ω_{S_0} is Cartier.
Since any fiber is Cartier, it follows that S_0 can only be singular along the singularities of the fibers. Three types of fibers may occur:

(F1) The general fiber is a smooth rational normal quartic C and S_0 is smooth along these fibers. Since any two general fibers are algebraically equivalent, we get $C^2 = 0$ and by the adjunction formula $C \cdot \omega_{S_0} = -2$.

(F2) A fiber might be a union of two conics meeting in one point, i.e. $C = C_1 + C_2$, where C_1 and C_2 are Weil divisors (not necessarily Cartier). The surface S_0 may be singular in the meeting point. We will see that this is at worst a rational double point in Lemma 4.2.4 below.

(F3) A fiber might be a double conic. This happens when S_0 is tangent to a surface which is a fiber of $X \longrightarrow B$. We can write $C \equiv 2C_0$ in $NE(S_0)$, where C_0 is a conic satisfying $C_0 \cdot \omega_{S_0} = -1$ (since $C \cdot \omega_{S_0} = -2$). It is easy to see that S_0 cannot be smooth along the whole double conic; indeed if it were, C_0 would be Cartier with $C_0^2 = 0$ and $C_0 \cdot \omega_{S_0} = -1$, which is impossible by the adjunction formula.

We say that the fiber is of type (F3a) if S_0 has isolated singularities along the conic (in which case it is normal there, since S_0 is Cohen-Macaulay) and of type (F3b) if S_0 is singular along the whole conic (in which case it is nonnormal there).

We will now take the normalization $\pi\colon T \longrightarrow S_0$ of S_0 and study this more closely. We will need to have a closer look at the fibers if type (F3b), i.e. the double conics along which S_0 is singular. We will concentrate on one fiber at the time.

Lemma 4.2.3. *Assume S_0 is not normal along one double fiber $C \equiv 2C_0$. Let $\pi\colon T \longrightarrow S_0$ be the normalization of S_0 along this fiber and denote by $\overline{D} \subset T$ its conductor (see below for the definition). Then T is fibered over B with fibers numerically equivalent to π^*C and $K_T = \pi^*\omega_{S_0} - \overline{D}$ with $K_T \cdot \pi^*C = -2$. Moreover we are in one of the following three cases:*

(FN1) $\pi^*C = C'$ and $\overline{D} = dC'$ for an integer $d \geq 1$, where C' is a smooth irreducible rational curve being mapped $2:1$ to C_0 by π with $C'^2 = 0$ and $C' \cdot K_T = -2$. Moreover K_T is Cartier.

(FN2) $\pi^*C = C_1' + C_2'$ *and* $\overline{D} = d_1C_1' + d_2C_2'$ *for two integers* $d_i \geq 1$*, where both* C_1' *and* C_2' *are smooth irreducible rational curves each being mapped* $1:1$ *to* C_0 *by* π*. Moreover both* $K_T + (d_1 - d_2)C_1'$ *and* $K_T + (d_2 - d_1)C_2'$ *are Cartier and* $C_1' \cap C_2' \neq \emptyset$*.*

(FN3) $\pi^*C = 2C_0'$ *and* $\overline{D} = dC_0'$ *for an integer* $d \geq 1$*, where* C_0' *is a smooth irreducible rational curve being mapped* $1:1$ *to* C_0 *by* π*. Moreover* $2K_T$ *is Cartier.*

Proof. Recall that ω_{S_0} is Cartier and

$$C_0 \cdot \omega_{S_0} = -1, \ C^2 = 0.$$

Let $D \subseteq S_0$ be the subscheme defined by the *conductor ideal* $\mathcal{I}_{D,S_0} := \mathrm{Ann}(\pi_*\mathcal{O}_T/\mathcal{O}_{S_0})$. Set-theoretically, D is the nonnormal locus of S_0, so it is supported on C_0. Let $\mathcal{I} := \pi^{-1}\mathcal{I}_{D,S_0} \cdot \mathcal{O}_T \subseteq \mathcal{O}_T$. This ideal defines a subscheme $\overline{D} \subseteq T$, the *conductor* of π. Since S_0 is Cohen-Macaulay, both D and $\overline{D}$ have pure dimension one, so in particular $\overline{D}$ is an effective Weil divisor (cf. [Re94, Proposition 2.2]). Moreover, by [Re94, Proposition 2.3],

$$\pi^*\omega_{S_0} \cong \mathcal{O}_T(K_T + \overline{D}). \tag{2.1.5}$$

Now both π^*C and $\overline{D}$ are effective Weil divisors supported on the inverse image of C_0, so we can write

$$\pi^*C = \sum_{i=1}^{r} c_iC_i', \qquad c_i \in \mathbb{Z}_{\geq 1},$$

$$\overline{D} = \sum_{i=1}^{r} d_iC_i', \qquad d_i \in \mathbb{Z}_{\geq 1},$$

where all the C_i' are prime divisors, i.e. irreducible curves on T.
Moreover, every C_i' is mapped by π to C_0. Let a_i be the degree of this map, i.e.

$$\pi_{|C_i'}: C_i' \xrightarrow{a_i:1} C_0.$$

Then we have

$$\pi_*C_i' = a_iC_0 \tag{2.1.6}$$

$$\pi_*\overline{D} = (\sum_{i=1}^{r} a_id_i)C_0.$$

We have by the projection formula and (2.1.6):

$$\begin{aligned} -2 &= C\cdot\omega_{S_0} = (\pi^*C)\cdot(\pi^*\omega_{S_0}) = (\sum_{i=1}^r c_iC_i')\cdot(\pi^*\omega_{S_0}) = \\ &= \sum_{i=1}^r c_i(\pi_*C_i')\cdot\omega_{S_0} = \sum_{i=1}^r c_ia_iC_0\cdot\omega_{S_0} = -\sum_{i=1}^r c_ia_i. \end{aligned}$$

This yields that $r=1$ or 2 and that we get the three possibilities $(c_1,a_1)=(1,2),(2,1)$ (for $r=1$) and $(c_1,a_1,c_2,a_2)=(1,1,1,1)$ (for $r=2$). Note that in any case we have

$$(\pi^*C)^2 = C^2 = 0 \text{ and } (\pi^*C)\cdot K_T = -2. \tag{2.1.7}$$

Case I: $r=1$, $(c_1,a_1)=(2,1)$: We have $\pi^*C=2C_1'$ and $\overline{D}=d_1C_1'$, so $2\overline{D}$ is Cartier, whence also $2K_T$ is Cartier by (2.1.5). The map $\pi_{|C_1'}\colon C_1'\longrightarrow C_0$ has degree one, whence C_1' is smooth of genus 0. It follows from (2.1.7) that $C_1'\cdot K_T=-1$. This is case (FN3).
Case II: $r=1$, $(c_1,a_1)=(1,2)$: We have $\pi^*C=C_1'$ and $\overline{D}=d_1C_1'$, so $\overline{D}$ is Cartier, whence also K_T is Cartier by (2.1.5). The map $\pi_{|C_1'}\colon C_1'\longrightarrow C_0$ has degree 2. It follows from (2.1.7) that $C_1'\cdot K_T=-2$. Since C_1' is Cartier it follows from the genus formula that $p_a(C_1')=0$, whence C_1' is smooth of genus 0. This is case (FN1).
Case III: $r=2$, $(c_1,a_1,c_2,a_2)=(1,1,1,1)$: We have $\pi^*C=C_1'+C_2'$ and $\overline{D}=d_1C_1'+d_2C_2'$, so both $\overline{D}-(d_1-d_2)C_1'=d_2(C_1'+C_2')$ and $\overline{D}-(d_2-d_1)C_2'=d_1(C_1'+C_2')$ are Cartier whence also $K_T+(d_1-d_2)C_1'$ and $K_T+(d_2-d_1)C_2'$ are Cartier by (2.1.5). The map $\pi_{|C_i'}\colon C_i'\longrightarrow C_0$ has degree 1 for both $i=1$ and 2, so the map on the fiber $\pi_{|\pi^*C}\colon\pi^*C\longrightarrow C_0$ has degree 2. It follows that both C_1' and C_2' are smooth of genus 0.
We now show that $C_1'\cap C_2'\neq\emptyset$. Assume the contrary; then, since T is Cohen-Macaulay and $\pi^*C=C_1'+C_2'$ is Cartier, T is smooth on this fiber, so C_1' and C_2' are Cartier as well. From here it follows that $\overline{D}$ and K_T are Cartier too. Since $g(C_i')=0$ we get by adjunction $K_T.C_i'=-2-C_i'^2$. On the other hand

$$K_T\cdot C_i' = (\pi^*\omega_{S_0}-d_1C_1'-d_2C_2')\cdot C_i' = C_0\cdot\omega_{S_0}-d_iC_i'^2 = -1-d_iC_i'^2,$$

whence $(d_i-1)C_i'^2=1$, so we must have $d_i=2$ and $C_i'^2=1$ and we get the contradiction $(\pi^*C)^2=C_1'^2+C_2'^2=2$.
This is case (FN2). *q.e.d.*

We will from now on work with the normal surface T, with fibers (FN1)-(FN3) over B. Since it is normal and Cohen-Macaulay it has isolated singularities and the only possible singularities lie on the singularities of the fibers. It follows that we have to look at the fibers of type (FN2) and (FN3).

Lemma 4.2.4. *Let $C' := \pi^*C = C'_1 + C'_2$ be a fiber of type (FN2) of T. Then $d_1 = d_2$ and T has at worst one A_n singularity along C'.*
Denote by $f\colon \tilde{T} \longrightarrow T$ a minimal resolution of such a singular point and let $E = \sum_{i=1}^n E_i$ be the exceptional divisor, where each E_i is a (-2)-curve and $\tilde{C}_i$ the strict transform of C'_i. Then both the $\tilde{C}_i$ are (-1)-curves, $f^*(C') = \tilde{C}_1 + \tilde{C}_2 + E_1 + \cdots + E_n$ and we have the following configuration:

$$\tilde{C}_1 \text{ —— } E_1 - - - E_n \text{ —— } \tilde{C}_2$$

Proof. If T is smooth along C', then both C'_1 and C'_2 are Cartier. Moreover $C'_1 \cdot C'_2 > 0$ by Lemma 4.2.3 and since $0 = C'_1 \cdot \pi^*C = C'_1 \cdot (C'_1 + C'_2) = C'^2_1 + C'_1 \cdot C'_2 > C'^2_1$ we get $C'^2_1 < 0$ and similarly $C'^2_2 < 0$. By Lemma 4.2.3 we have $-2 = K_T \cdot (C'_1 + C'_2)$ so the only possibility by the adjunction formula is $C_1'^2 = C_2'^2 = K_T \cdot C'_1 = K_T \cdot C'_2 = -1$ and $C'_1 \cdot C'_2 = 1$. Since $-1 = K_T \cdot C'_1 = (\pi^*\omega_{S_0} - d_1 C'_1 - d_2 C'_2) \cdot C'_1 = \omega_{S_0} \cdot C_0 + d_1 - d_2 = -1 + d_1 - d_2$ we get $d_1 = d_2$.
Assume now that T has $r > 0$ singular points along the fiber C'. Let $f\colon \tilde{T} \longrightarrow T$ be a minimal resolution of these singular points with exceptional divisor $E = \sum_{j=1}^{n_1} E_{1j} + \cdots + \sum_{j=1}^{n_r} E_{rj}$.
Since the question is local we can assume that $\tilde{T}$ is smooth.
Denote by $\tilde{C}_i$ the strict transform of C'_i. Recall that by Lemma 4.2.3 $\tilde{C}_i$ is a smooth rational curve. Moreover $K_T = \pi^*\omega_{S_0} - d_1 C'_1 - d_2 C'_2$ and $K_T + (d_1 - d_2)C'_1$ is Cartier. Without loss of generality we can assume $d_1 \geq d_2$.
We have

$$K_{\tilde{T}} \equiv f^*(K_T + (d_1 - d_2)C'_1) - (d_1 - d_2)\tilde{C}_1 - \sum \alpha_{ij} E_{ij}, \quad \alpha_{ij} \in \mathbb{Z}_{\geq 0} \tag{2.1.8}$$

$$f^*(C') \equiv \tilde{C}_1 + \tilde{C}_2 + \sum \beta_{ij} E_{ij}, \quad \beta_{ij} \in \mathbb{Z}_{>0}. \tag{2.1.9}$$

(The fact that all $\beta_{ij} > 0$ follows for instance from [KoMo, Lemma 2.27].) Moreover, by e.g. [Mat, Theorem 4-6-2] or [KoMo, Corollary 4.3], we have that for fixed i either all $\alpha_{ij} = 0$ or all $\alpha_{ij} > 0$.
We have, using (2.1.9),

$$0 = \tilde{C}_1 \cdot f^*(C'_1 + C'_2) = \tilde{C}_1^2 + \tilde{C}_1 \cdot \tilde{C}_2 + \tilde{C}_1 \cdot \sum \beta_{ij} E_{ij} > \tilde{C}_1^2,$$

and similarly for $\tilde{C}_2$, whence by the adjunction formula

$$\tilde{C}_i^2 \leq -1 \text{ and } \tilde{C}_i \cdot K_{\tilde{T}} \geq -1 \text{ for } i = 1, 2. \tag{2.1.10}$$

Moreover

$$f^*(C') \cdot K_{\tilde{T}} = -2.$$

Since f is a minimal resolution and all the E_{ij} are contracted, we have $E_{ij}^2 \leq -2$, whence $E_{ij} \cdot K_{\tilde{T}} \geq 0$ by the adjunction formula for all i, j. Using (2.1.9) and (2.1.10) we therefore have

$$-2 = f^*(C') \cdot K_{\tilde{T}} = K_{\tilde{T}} \cdot (\tilde{C}_1 + \tilde{C}_2 + \sum \beta_{ij} E_{ij}) \geq -2 + \sum \beta_{ij} E_{ij} \cdot K_{\tilde{T}},$$

which yields

$$\tilde{C}_1^{\ 2} = \tilde{C}_2^{\ 2} = \tilde{C}_1 \cdot K_{\tilde{T}} = \tilde{C}_2 \cdot K_{\tilde{T}} = -1 \tag{2.1.11}$$

and

$$E_{ij} \cdot K_{\tilde{T}} = 0, \ E_{ij}^2 = -2 \text{ forall } i, j.$$

From (2.1.9) and (2.1.11) we immediately get

$$0 = \tilde{C}_1 \cdot f^*(C') = -1 + \tilde{C}_1 \cdot \tilde{C}_2 + \sum \beta_{ij} E_{ij} \cdot \tilde{C}_1,$$

and similarly for $\tilde{C}_2$ whence

$$\tilde{C}_1 \cdot \tilde{C}_2 = 0 \tag{2.1.12}$$

and

$$\sum \beta_{ij} E_{ij} \cdot \tilde{C}_1 = \sum \beta_{ij} E_{ij} \cdot \tilde{C}_2 = 1. \tag{2.1.13}$$

By the projection formula we have

$$\begin{aligned} \tilde{C}_i \cdot f^*(K_T + (d_1 - d_2)C_1') \ &= \ C_i' \cdot (K_T + (d_1 - d_2)C_1') \ = \\ = \ C_i' \cdot (\pi^*\omega_{S_0} - d_1 C_1' - d_2 C_2' + (d_1 - d_2)C_1') \ &= \ C_i' \cdot (\pi^*\omega_{S_0} - d_2(C_1' + C_2')) \ = \\ = \ C_i' \cdot \pi^*(\omega_{S_0} - d_2 C) \ &= \ C_0 \cdot (\omega_{S_0} - d_2 C) \ = \ -1. \end{aligned}$$

Combining (2.1.8) and (2.1.11) yields

$$\begin{aligned} -1 \ = \ \tilde{C}_1 \cdot K_{\tilde{T}} \ &= \ \tilde{C}_1 \cdot (f^*(K_T + (d_1 - d_2)C_1') - (d_1 - d_2)\tilde{C}_1 - \sum \alpha_{ij} E_{ij}) \ = \\ &= \ -1 + (d_1 - d_2) - \sum \alpha_{ij} E_{ij} \cdot \tilde{C}_1 \end{aligned}$$

whence

$$\alpha_{ij} E_{ij} \cdot \tilde{C}_1 = d_1 - d_2 \tag{2.1.14}$$

Arguing in the same way for $\tilde{C}_2$ yields

$$\alpha_{ij} E_{ij} \cdot \tilde{C}_2 = 0. \tag{2.1.15}$$

By Lemma 4.2.3 we have $C_1' \cap C_2' \neq \emptyset$, so by (2.1.12) we have that C_1' and C_2' have at least one singularity of T in common. By (2.1.13) and the fact that all the $\beta_{ij} > 0$, it follows that T has only one singular point on the fiber C', so we have $E = \sum_{j=1}^{r} E_{1j}$. From (2.1.15) it immediately follows that all the $\alpha_{1j} = 0$, whence $d_1 = d_2$ by (2.1.14). From (2.1.13) we can assume without loss of generality that

$$\tilde{C}_1 \cdot E_{11} = \beta_{11} = \tilde{C}_2 \cdot E_{1r} = \beta_{1r} = 1$$

and that $\tilde{C}_1 \cdot E_{1j} = 0$ for $j \geq 2$ and $\tilde{C}_2 \cdot E_{1j} = 0$ for $j \leq r-1$.
From (2.1.9) we have $0 = E_{12} \cdot f^*(C') = 1 - 2 + E_{12} \cdot \sum_{j=3}^{r} \beta_{1j} E_{1j}$, whence $E_{12} \cdot \sum_{j=3}^{r} \beta_{1j} E_{1j} = 1$ and similarly $E_{1r} \cdot \sum_{j=3}^{r} \beta_{1j} E_{1j} = 1$. Proceeding inductively it is easy to see that all the $\beta_{1j} = 1$ and that $E = \sum_{j=1}^{r} E_{1j}$ is a chain of (-2)-curves with one of the end tails intersecting $\tilde{C}_1$ in one point and the other intersecting $\tilde{C}_2$ in one point.

q.e.d.

As we showed above for S_0 it immediately follows that T cannot be smooth along a fiber of type (FN3).

Lemma 4.2.5. *Let $C' = 2C_0'$ be a fiber of type (FN3) of T. Then the singular locus of T along C_0 consists of either one or two rational double points. Denote by $f\colon \tilde{T} \longrightarrow T$ the desingularization of these points and let E be the exceptional divisor and $\tilde{C}_0$ the strict transform of C_0'. Then $\tilde{C}_0$ is a (-1)-curve and one of the three following cases occurs:*

(a) T has two A_1 singularities along C_0', $E = E_1 + E_2$ and $f^*(C') = 2\tilde{C}_0 + E_1 + E_2$, where E_1 and E_2 are (-2)-curves with the following configuration:

$$\begin{array}{ccc} \tilde{C}_0 & \text{——} & E_1 \\ | & & \\ E_2. & & \end{array}$$

(b) T has one A_3 singularity along C_0', $E = E_1 + E_2 + E_3$ and $f^*(C') = 2\tilde{C}_0 + 2E_1 + E_2 + E_3$, where E_1, E_2 and E_3 are (-2)-curves with the following configuration:

$$\begin{array}{ccccc} \tilde{C}_0 & \text{——} & E_1 & \text{——} & E_2 \\ & & | & & \\ & & E_3. & & \end{array}$$

(c) *T has one D_n singularity along C_0', for $n \geq 4$, $E = E_1 + \cdots + E_n$ and $f^*(C') = 2\tilde{C}_0 + 2E_1 + 2E_2 + \ldots + 2E_{n-2} + E_{n-1} + E_n$, where all the E_i are (-2)-curves with the following configuration:*

$$\begin{array}{ccccccc} \tilde{C}_0 & \text{——} & E_1 - - - & E_{n-2} & \text{——} & E_{n-1} \\ & & & | & & \\ & & & E_n. & & \end{array}$$

Proof. Let $f\colon \tilde{T} \longrightarrow T$ be the desingularization of the $r \geq 1$ singular points of T on C_0' with exceptional divisor $E = \sum_{j=1}^{n_1} E_{1j} + \cdots + \sum_{j=1}^{n_r} E_{rj}$. The question is local so we can assume that $\tilde{T}$ is smooth.
We have

$$f^*K_T = K_{\tilde{T}} + \sum_{j=1}^{n_1} \alpha_{1j} E_{1j} + \cdots + \sum_{j=1}^{n_r} \alpha_{rj} E_{rj}, \qquad 2\alpha_{ij} \in \mathbb{Z}_{\geq 0} \tag{2.1.16}$$

(recall that $2K_T$ is Cartier by Lemma 4.2.3) and

$$\text{for fixed } i \text{ either all } \alpha_{ij} = 0 \text{ or all } \alpha_{ij} > 0 \tag{2.1.17}$$

(see e.g. [Mat, Theorem 4-6-2] or [KoMo, Corollary 4.3]). Moreover

$$f^*C' = 2\tilde{C}_0 + \sum_{j=1}^{n_1} \beta_{1j} E_{1j} + \cdots + \sum_{j=1}^{n_r} \beta_{rj} E_{rj}, \qquad \beta_{ij} \in \mathbb{Z}_{>0}. \tag{2.1.18}$$

(The fact that all $\beta_{ij} > 0$ follows for instance from [KoMo, Lemma 2.27]).
Now if one of the singular points of T is not a rational singularity, the second case of (2.1.17) has to occur for $i = 1$ say, i.e. all $\alpha_{1j} > 0$. Multiplying (2.1.16) with $\tilde{C}_0$ and using the projection formula together with $-K_T \cdot C_0' = 1$, we find that $\tilde{C}_0 \cdot K_{\tilde{T}} < -1$, whence by adjunction $\tilde{C}_0^{\,2} \geq 0$. From (2.1.18) we get

$$0 = C' \cdot C_0' = (f^*C') \cdot \tilde{C}_0 = 2\tilde{C}_0^{\,2} + \tilde{C}_0 \cdot (\sum \beta_{ij} E_{ij}), \tag{2.1.19}$$

whence $\tilde{C}_0 \cdot E_{ij} = 0$ for all ij, a contradiction.
So $f^*K_T = K_{\tilde{T}}$ and $f\colon \tilde{T} \longrightarrow T$ is a rational resolution of singularities. It immediately follows that $\tilde{C}_0$ is a (-1)-curve and each E_{ij} is a (-2)-curve. Moreover from (2.1.19) we get

$$\tilde{C}_0 \cdot (\sum_{j=1}^{n_1} \beta_{1j} E_{1j} + \cdots + \sum_{j=1}^{n_r} \beta_{rj} E_{rj}) = 2, \tag{2.1.20}$$

whence $r \leq 2$.
We treat the two cases $r = 1$ and $r = 2$ separately.
Case I: $r = 2$. We can write

$$f^*C' = 2\tilde{C}_0 + \sum_{i=1}^{m} a_i E_i + \sum_{j=1}^{n} b_j F_j, \qquad a_i, b_j \in \mathbb{Z}_{\geq 1}, \tag{2.1.21}$$

with all the E_i and F_j being (-2)-curves. From (2.1.20) we get (after possibly renumbering the indices)

$$\tilde{C}_0 \cdot E_1 = \tilde{C}_0 \cdot F_1 = a_1 = b_1 = 1 \text{ and } \tilde{C}_0 \cdot E_i = \tilde{C}_0 \cdot F_j = 0 \text{ for all } i \geq 2, j \geq 2. \tag{2.1.22}$$

Using (2.1.21) we easily compute $E_1 \cdot \sum_{i=2}^{m} a_i E_i = 0$ and $F_1 \cdot \sum_{j=2}^{n} b_j F_j = 0$, and by connectedness of the exceptional divisors we must be in case (a).
Case II: $r = 1$. We can write

$$f^*C' = 2\tilde{C}_0 + \sum_{i=1}^{m} a_i E_i, \qquad a_i \in \mathbb{Z}_{\geq 1}, \tag{2.1.23}$$

where all the E_i are (-2)-curves. From (2.1.20) we get (after possibly renumbering the indices) three subcases:
Case IIa: $\tilde{C}_0 \cdot E_1 = 2$. It follows that $a_1 = 1$ and $\tilde{C}_0 \cdot E_i = 0$ for all $i \geq 2$. From (2.1.23) we easily compute $E_1 \cdot \sum_{i=2}^{m} a_i E_i = -2$, a contradiction.
Case IIb: $\tilde{C}_0 \cdot E_1 = \tilde{C}_0 \cdot E_2 = 1$. It follows that $a_1 = a_2 = 1$ and $\tilde{C}_0 \cdot E_i = 0$ for all $i \geq 3$. From (2.1.23) we easily compute $E_1 \cdot \sum_{i=2}^{m} a_i E_i = E_2 \cdot \sum_{i=3}^{m} a_i E_i = 0$, a contradiction on the connectedness of the exceptional divisor.
Case IIc: $\tilde{C}_0 \cdot E_1 = 1$. It follows that $a_1 = 2$ and $\tilde{C}_0 \cdot E_i = 0$ for all $i \geq 2$. From (2.1.23) we easily compute $E_1 \cdot \sum_{i=2}^{m} a_i E_i = 2$. After renumbering we can assume that either $a_2 = a_3 = 1$, $E_1 \cdot E_2 = E_1 \cdot E_3 = 1$ and $E_1 \cdot E_j = 0$ for all $j \geq 4$, or $a_2 = 2$, $E_1 \cdot E_2 = 1$ and $E_1 \cdot E_j = 0$ for all $j \geq 3$. The first case yields case (b) above and the second case yields, after iterating, case (c) above. *q.e.d.*

Summarising, we have proved the following:

Proposition 4.2.6. *With the same assumptions as in (2.1.2) and (2.1.3), assume furthermore that S_0 has $l_0 \in \mathbb{Z}_{\geq 0}$ degenerate fibers of types (F2) or (F3a).*
Denote by $f\colon \tilde{S}_0 \longrightarrow S_0$ a minimal resolution. Then $\tilde{S}_0$ has exactly $l \geq l_0$ reducible fibers over B, which are of one of the following three types:

(i) $\tilde{C}_1 + \tilde{C}_2 + E_1 + \cdots + E_r$, $r \geq 0$, *the* $\tilde{C}_i$ *are* (-1)*-curves and the* E_j *are* (-2)*-curves, with the following configuration:*

$$\tilde{C}_1 \text{ --- } E_1 - - - E_r \text{ --- } \tilde{C}_2$$

(ii) $2\tilde{C}_0 + E_1 + E_2$, *where* $\tilde{C}_0$ *is a* (-1)*-curve and* E_1 *and* E_2 *are* (-2)*-curves with the following configuration:*

$$\begin{array}{ccc} \tilde{C}_0 & \text{——} & E_1 \\ | & & \\ E_2. & & \end{array}$$

(iii) $2\tilde{C}_0 + 2E_1 + E_2 + E_3$, *where* $\tilde{C}_0$ *is a* (-1)*-curve and* E_1, E_2 *and* E_3 *are* (-2)*-curves with the following configuration:*

$$\begin{array}{ccccc} \tilde{C}_0 & \text{——} & E_1 & \text{——} & E_2 \\ & & | & & \\ & & E_3. & & \end{array}$$

(iv) $2\tilde{C}_0 + 2E_1 + 2E_2 + \ldots + 2E_{r-2} + E_{r-1} + E_r$, $r \geq 4$, *where* $\tilde{C}_0$ *is a* (-1)*-curve and all the* E_i *are* (-2)*-curves with the following configuration:*

$$\begin{array}{ccccccc} \tilde{C}_0 & \text{——} & E_1 & \text{- - -} & E_{n-2} & \text{——} & E_{n-1} \\ & & & & | & & \\ & & & & E_n. & & \end{array}$$

Moreover $K_{\tilde{S}_0} \equiv f^*\omega_{S_0} - \frac{m}{2}\tilde{C}$ *for some* $m \in \mathbb{Z}_{\geq 0}$, *where* $\tilde{C}$ *is the numerical equivalence class of the fiber.*

Proof. Take first the normalization $\pi \colon T \longrightarrow S_0$. As seen above this is an isomorphism outside of the fibers of S_0 of type (F3b). Let l be the number of degenerate fibers of T. As seen above we have $l \geq l_0$ (the difference is the numbers of fibers of type (F3b) of S_0 which have not turned smooth under the normalization).
Now we let $f' \colon \tilde{S}_0 \longrightarrow T$ be a minimal resolution. It follows from Lemmas 4.2.4 and 4.2.5 that the resolution is rational, so $K_{\tilde{S}_0} = f'^* K_T$ and that the fibers are as in (i)-(iv). Moreover, by Lemma 4.2.3 we get $K_{\tilde{S}_0} = f'^* K_T \equiv f'^*(\pi^*\omega_{S_0} - \frac{m}{2}C') = f^*\omega_{S_0} - \frac{m}{2}\tilde{C}$ as claimed. *q.e.d.*

Corollary 2.1.24. *The minimal model* R *of* $\tilde{S}_0$ *is a ruled surface over* B *satisfying* $K_R^2 = 8(1 - g(B)) \geq K_{\tilde{S}_0}^2 + l_0$.

Proof. To reach the minimal model we succesively blow down (-1)-curves in each of the l degenerate fibers of $\tilde{S}_0$ until we reach a ruled surface. Note that when we blow down a (-1)-curve any (-2)-curve intersecting it in one point becomes a (-1)-curve, so we can continue the process until we reach an irreducible fiber.
It immediately follows that $K_R^2 \geq K_{\tilde{S}_0}^2 + l \geq K_{\tilde{S}_0}^2 + l_0$, since every blow down raises the self-intersection of the canonical bundle by one. *q.e.d.*

4.3 Bounding the degree of uniruled 3-folds

As said in the introduction, it is possible to associate a Mori fiber space to a uniruled 3-fold. Moreover in the special case of pairs $(X, \mathcal{H})$ consisting of an irreducible terminal $\mathbb{Q}$-factorial 3-fold X and a big uniruled system $\mathcal{H}$ on X, all the possible Mori fiber spaces are classified in 4.1.9.
A problem that seems natural to investigate at this point is the following: finding conditions under which a 3-fold of this kind is uniruled of degree one with respect to the polarization.
We find easy bounds (4.3.1, 4.3.2 and 4.3.3), which are not sharp.
We start with the following:

Proposition 4.3.1. *Let $(X, \mathcal{H})$ be a pair consisting of an irreducible terminal $\mathbb{Q}$-factorial 3-fold X and a base point free big line bundle $\mathcal{H}$ on X. Set $d := \mathcal{H}^3$ and $n := h^0(X, \mathcal{H}) - 1 \geq 4$.*
If $d < \frac{4}{3}n - \frac{4}{3}$, then X is uniruled of $\mathcal{H}$-degree one, except when a $\sharp$-minimal model of $(X, \mathcal{H})$ is $(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))$, $(\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(2))$ or $(X_4 \subset \mathbb{P}(1,1,1,1,2), X_4 \cap \{x_0 = 0\})$.

Proof. We claim that $(X, \mathcal{H})$ is a pair with a big uniruled system and $\rho(X, \mathcal{H}) < 2/3$. Indeed let $H \in |\mathcal{H}|$ be general (whence smooth, since it does not pass through the singular points of X) and C the general smooth curve section of X.
From the genus formula on the surface H and by adjunction we get

$$2g(C) - 2 = C \cdot (2H + K_X). \qquad (3.1)$$

Since $n \geq 4$ we have $2n - 4 \geq \frac{4}{3}n - \frac{4}{3}$, whence $d < 2n - 4$. Set $\mathcal{H}_C := \mathcal{H} \otimes \mathcal{O}_C$. Then $h^0(\mathcal{H}_C) \geq n - 1$ and $\mathrm{Cliff}\mathcal{H}_C = d - 2(h^0(\mathcal{H}_C) - 1) \leq d - 2(n-2) < 0$. By Clifford's theorem $h^1(\mathcal{H}_C) = 0$ and by Riemann-Roch $g(C) = d - h^0(\mathcal{H}_C) + 1 \leq d - n + 2$, which together with (3.1) and the assumption on d yields

$$C \cdot \left(\frac{3}{2}H + K_X\right) = 2g(C) - 2 - \frac{1}{2}C \cdot H \leq \frac{3}{2}d - 2n + 2 < 0.$$

This shows that $\rho < 2/3$. At the same time we also get

$$H^2 \cdot \Big(H + K_X\Big) = C \cdot \Big(\frac{3}{2}H + K_X\Big) - \frac{1}{2}C \cdot H < 0,$$

so by the adjunction formula H is a smooth surface of negative Kodaira dimension, which shows that $(X, \mathcal{H})$ is a pair with a big uniruled system.
It follows that the $\sharp$-Minimal Model $(X^\sharp, \mathcal{H}^\sharp)$ is in the list of Theorem 4.1.9 and moreover it cannot be as in (II) since $\rho(X, \mathcal{H}) = 2/3$ in this case. In the cases (III)-(V) one immediately sees that $(X^\sharp, \mathcal{H}^\sharp)$ is uniruled of $\mathcal{H}^\sharp$-degree one. Note that $d^\sharp = d$ and $n^\sharp \geq n$. By using the table above we see that the cases in (I) with $d^\sharp < \frac{4}{3}n^\sharp - \frac{4}{3}$ which are not uniruled of $\mathcal{H}^\sharp$-degree one are the cases (vi) with $a = 2$, (viii) with $b = 2$ and (xi). *q.e.d.*

We have the following corollary:

Corollary 4.3.2. *Let $X \subset \mathbb{P}^n$ be a nondegenerate irreducible 3-dimensional variety of degree d.*
If $d < \frac{4}{3}n - \frac{4}{3}$ then X is uniruled by lines, except when a $\sharp$-minimal model of $(X, \mathcal{O}_X(1))$ is $(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))$ or $(\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(2))$.

Proof. If $n \geq 4$, let $\pi \colon T \longrightarrow X$ be a resolution of the singularities of X and $\mathcal{H} := \pi^*\mathcal{O}_X(1)$. Then $\mathcal{H}^3 = d$ and $h^0(\mathcal{H}) \geq n + 1$ and we apply Proposition 4.3.1.
Since $\mathcal{H}$ is birational the case $(X_4 \subset \mathbb{P}(1, 1, 1, 1, 2), X_4 \cap \{x_0 = 0\})$ from Proposition 4.3.1 does not occur as $\sharp$-minimal model of $(X, \mathcal{H})$, since in this case the map associated to the system is not birational, and the result follows. *q.e.d.*

It is interesting to note that similar bounds as in Proposition 4.3.1 can be obtained by using results of Horowitz [Ho83] and Reid [Re86] respectively. As an example we briefly show the bound one would get using Horowitz's result. Note that the obtained bound is stronger than Proposition 4.3.1 for $n \geq 29$ but one has to assume that the 3-fold is smooth and the line bundle is very ample.

Proposition 4.3.3. *Let X be a smooth 3-fold and $\mathcal{H}$ a very ample line bundle on X with $d := \mathcal{H}^3$ and $n := h^0(\mathcal{H}) - 1$. If $n \geq 12$ and $d < \frac{3}{2}(n - 4)$, or $7 \leq n \leq 11$ and $d < \frac{4}{3}(n - 3)$, then X is uniruled of $\mathcal{H}$-degree one.*

Proof. By [Ho83, Corollary p. 668], if $d < \frac{3}{2}(n - 4)$ then X is ruled by planes or quadrics and if $d < \frac{4}{3}(n - 3)$ then X is ruled by planes, in which cases it is clearly uniruled by lines. Now we note that $\frac{4}{3}(n - 3) > \frac{3}{2}(n - 4)$ for $n \leq 11$ and that the condition $d \geq n - 2$ requires $n \geq 7$. *q.e.d.*

4.3.4. By [Me02, Proof of Theorem 5.8], if $d < 2n - 4$ then a $\sharp$-minimal model $(X^\sharp, \mathcal{H}^\sharp)$ of $(X, \mathcal{H})$ is in the list of Theorem 4.1.9. At the end of the first section we saw that, up to some exception, the 3-folds in case (I) of Theorem 4.1.9 satisfy $d \geq 2n - 10$. Moreover example 4.2.2 shows that $d = 2n - 10$ can happen for 3-folds in case (II) of Theorem 4.1.9; therefore we want to prove that in this case we have $d \geq 2n - 10$, i.e. that the following result holds:

Conjecture 4.3.5. *Let $(X, \mathcal{H})$ be a pair consisting of an irreducible terminal $\mathbb{Q}$-factorial 3-fold X and a base point free big line bundle $\mathcal{H}$ on X. Set $d := \mathcal{H}^3$ and $n := h^0(X, \mathcal{H}) - 1$. If $d < 2n - 10$, then X is uniruled of $\mathcal{H}$-degree one, except when a $\sharp$-minimal model of $(X, \mathcal{H})$ is $(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))$ (in which case $d = 27$ and $n = 19$) or $(\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(2))$ (in which case $d = 16$ and $n = 14$).*

By the study of singularities in the second section of this chapter, we saw that, starting with a section S as in 2.1.2, we can reach a new surface R such that $K_R^2 = 8(1 - g(B))$, the difference between K_R^2 and K_S^2 being the number of contracted curves (2.1.24). We think that we could conclude that $d \geq 2n - 10$ if we prove that the general member of $\mathcal{H}$ is a smooth surface fibered over B with $\geq \frac{n-5}{2}$ fibers which are unions of two conics intersecting in one point (the other fibers are smooth quartics). We note that both equalities are obtained in example 4.2.2.

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